Math 475, Fall 2015
Homework 11
Due: **Monday**, Dec. 14

(1) Show that every red-blue coloring of the edges of $K_{3,3}$ either has a red path of length 3 or a blue path of length 3.

(2) (a) Show that every red-blue coloring of the edges of $K_7$ either has a red triangle or a blue 4-cycle.
(b) Give an example of a red-blue coloring of the edges of $K_6$ which has no red triangle and no blue 4-cycle.

(3) Prove the following inequality for the generalized Ramsey number:

$$R(s_1, s_2, \ldots, s_k) \le \frac{(s_1 + s_2 + \cdots + s_k - k)!}{(s_1 - 1)!(s_2 - 1)! \cdots (s_k - 1)!}.$$

(4) Let $r$ be a positive integer. A red-blue coloring of the $r$-subsets of $[n]$ is a choice of red or blue for each subset of size $r$. Given a coloring and $k \ge r$, a subset $S \subseteq [n]$ with $|S| = k$ is red (respectively, blue) if all of its $r$-element subsets are red (respectively, blue). Given integers $k, \ell \ge r$, let $R_r(k, \ell)$ be the smallest integer $n$ (if it exists) such that if $m \ge n$, then any red-blue coloring of the $r$-subsets of $[m]$ either has a red subset of size $k$ or a blue subset of size $\ell$. (If $r = 2$, then $R_2(k, \ell) = R(k, \ell)$.)

Use a double induction (first on $r$, second on $k + \ell$) to show that the numbers $R_r(k, \ell)$ always exist, and that if $r \ge 2$ and $k, \ell > r$, then

$$R_r(k, \ell) \le R_{r-1}(R_r(k - 1, \ell), R_r(k, \ell - 1)) + 1.$$

[**Hint:** When $r = 2$, this inequality is the one we proved for Ramsey numbers. Here's how to generalize that argument. Let $n$ be the right side of the inequality. Pick a red-blue coloring of the $r$-subsets of $[n]$. Define a red-blue coloring of the $(r - 1)$-subsets on $[n - 1]$ by letting the color of $S \subseteq [n - 1]$ be the color of $S \cup \{n\}$ in the original coloring.]

(5) The goal of this exercise is to prove that $R(3, 5) = 14$.
(a) Show that $R(3, 5) \le 14$.
(b) Construct a red-blue coloring of the edges of $K_{13}$ as follows. The vertices are the numbers $\{0, 1, \ldots, 12\}$. Color the following edges red[1]

$$\{i, i + 1\} \quad (\text{for } 0 \le i \le 11), \qquad \{i, i + 5\} \quad (\text{for } 0 \le i \le 7),$$
$$\{i, i + 8\} \quad (\text{for } 0 \le i \le 4), \qquad \{0, 12\}.$$

All other edges are blue.
Show that there is no red triangle or blue $K_5$. In particular, $R(3, 5) \ge 14$.
[**Hint:** This is the same as saying there is no choice of $0 \le a < b < c \le 12$ such that $b - a, c - b, c - a \in \{1, 5, 8, 12\}$ and no choice of $0 \le a < b < c < d < e \le 12$ such that all of the differences are in $\{2, 3, 4, 6, 7, 9, 10, 11\}$. For the second claim, note that the sum of all 4 differences is $e - a$ must be $\ge 8$ since the smallest value of a difference is 2.]

[1] Alternatively: the vertices are the integers modulo 13 and $\{i, j\}$ is a red edge if and only if there is a solution in $x$ to $x^3 \equiv i - j \pmod{13}$. You may use this if you like, but you're free to ignore this fact.