Deletion-contraction and chromatic polynomials
Math 475
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1. Deletion-contraction

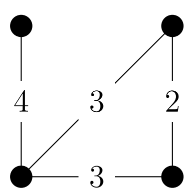
Let G be a graph and e an edge of G. There are two important operations (deletion and contraction) that we can perform on G using e and which are useful for certain kinds of induction proofs.

The **deletion** of e is denoted $G \setminus e$ and is a graph with the same vertices as G, and the same edges, except we don't use e.

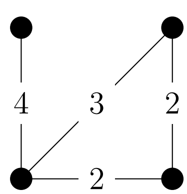
The **contraction** of e is denoted G/e. Let $e = \{x, y\}$. To define it, take the vertices of G, replace the two vertices x, y with a single vertex that we will call z. For each edge in G that does not use x or y, add it into G/e. For each vertex a different from x and y, the number of edges between a and z in G/e is the number of edges between a and x plus the number of edges between a and y.

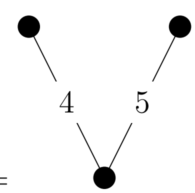
To visualize this, pretend we are shrinking e until x and y become the same point (hence the use of the word contraction). However, this is slightly misleading: if there were another edge between x and y, it would end up becoming a loop at z, but we don't take these into consideration. To be more accurate, we would have to allow graphs to have loops, but this creates a lot of notational headaches, so we throw them away whenever possible.

Here's a small example to illustrate. Say our graph is as follows (I put numbers on the edges to denote multiple edges):

$G =$ 

Let e be one of the edges between the bottom two vertices. Then

$G \setminus e =$ 

$G/e =$ 

Visually, G/e is the result of shrinking the bottom edge of G towards its midpoint. As we said before, the other two bottom edges would end up becoming loops on the bottom, but we remove them.

2. Spanning trees

Let $\tau(G)$ (that letter is TAU) be the number of spanning trees of G.

Proposition 2.1. $\tau(G) = \tau(G \setminus e) + \tau(G/e)$.

Proof. Write $e = \{x, y\}$. $\tau(G \setminus e)$ counts the number of spanning trees in G that do not use the edge e while $\tau(G/e)$ counts the number of spanning trees in G that do use the edge e.

The second requires some more explanation: if we have a spanning tree T of G that uses e and we contract e, the remaining edges of T become a spanning tree of $G \setminus e$. We can reverse this: by the way we defined it, there is a bijection between the edges of $G \setminus e$ and the edges of G whose endpoints aren't $\{x, y\}$ (because we discarded the loops). So if we have a spanning tree of $G \setminus e$, take the corresponding edges of T and add e to get a spanning tree of G. We just defined a bijection between spanning trees of G using e and spanning trees of G/e. One thing to note: spanning trees never use more than one edge with the same endpoints, and never use loops; since edges with the same endpoints as e correspond to loops in G/e, it's okay that we discarded them in our definition, although we also see that it really makes no difference if we keep them around or not.

Every spanning tree of G either uses e or doesn't, so we get the desired identity. □

We also proved in class that if we order the vertices so that e is an edge between the first two vertices, then $\det(L_G[1]) = \det(L_{G\setminus e}[1]) + \det(L_{G/e}[1])$, where L_G is the Laplacian matrix of G, and the [1] means "delete the first row and first column of the matrix". Combining this with the recursion for τ, we proved the matrix-tree theorem: $\tau(G) = \det(L_G[1])$. I won't recall the details here.

3. Chromatic polynomials

If G is a graph, and $k \geq 0$ is a non-negative integer, a **proper k-coloring** is a way to label the vertices of G with the numbers (colors) $\{1, \ldots, k\}$ so that two vertices that are connected by an edge have different labels. We are free to use colors multiple times and we don't have to use all of them. Let $\chi_G(k)$ (that letter is CHI) be the number of ways to properly color the vertices with k colors. The **chromatic number** of G, denoted $\chi(G)$, is the smallest k such that G has a proper k-coloring.

Lemma 3.1. *Let x, y be two vertices of G with exactly one edge e between them. Then*

$$\chi_G(k) = \chi_{G\setminus e}(k) - \chi_{G/e}(k).$$

Proof. By the definitions, a proper k-coloring of G is the same thing as a proper k-coloring of $G \setminus e$ where x and y get different labels. On the other hand, proper k-colorings of $G \setminus e$ where x and y receive the same color are naturally in bijection with proper k-colorings of G/e: if z is the result of contracting x and y, make its color the common color of x and y. The identity $\chi_G(k) = \chi_{G\setminus e}(k) - \chi_{G/e}(k)$ is a translation of what we just said: proper k-colorings of G are the same thing as proper k-colorings of $G \setminus e$ once we subtract off all of those that give x and y the same color. □

The assumption about x and y having exactly one edge between them is a little bit annoying, but it's easy to get around. Let G be a graph. Construct a simple graph $\overline{G}$ whose vertices are the same as G and where x and y have an edge in $\overline{G}$ if they have at least one edge in G. In other words, multiple edges in G get replaced by a single edge in $\overline{G}$.

Lemma 3.2. $\chi_G(k) = \chi_{\overline{G}}(k)$.

Proof. The definition of proper k-coloring only involves labeling vertices and the conditions on them only depend on whether or not two vertices have the same color if they're connected by an edge (but we don't care how many edges). □

Now we're ready to prove the main result:

Theorem 3.3. *If G is a graph with n vertices, then $\chi_G(k)$ is a polynomial in k of degree n (more precisely, there is a unique polynomial of degree n whose values agree with $\chi_G(k)$ at all non-negative integer inputs k).*

Proof. By Lemma 3.2, it is enough to prove this for simple graphs G. We proceed by induction on the number of edges. If there are no edges in G, then any labeling of the vertices is a proper k-coloring, so $\chi_G(k) = k^n$ which is certainly a polynomial of degree n.

Now assume we've proved this for graphs with $< m$ edges and let G be a graph with m edges. Let e be an edge of G. Then $G \setminus e$ and G/e both have $< m$ edges. So $\chi_{G\setminus e}(k)$ is a polynomial in k of degree n and G/e is a polynomial in k of degree $n-1$. By Lemma 3.1, $\chi_G(k) = \chi_{G\setminus e}(k) - \chi_{G/e}(k)$, so $\chi_G(k)$ is a polynomial in k of degree n. □

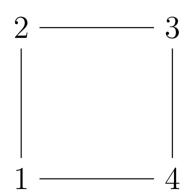
For notation, we will write $\chi_G(z)$ for this polynomial (z is now a variable) and we will use k to denote non-negative integers. This is the **chromatic polynomial** of G. Then the chromatic number is the smallest positive integer k such that $\chi_G(k) \neq 0$.

Here are some easy properties:

Proposition 3.4. (1) *G has at least one vertex if and only if $\chi_G(0) = 0$.*
(2) *G has at least one edge if and only if $\chi_G(1) = 0$. (The converse is clearly true.)*
(3) *If G has an odd length cycle, then $\chi_G(2) = 0$. (The converse is also true, as we will see when we discuss bipartite graphs.)*

How about a property that determines if $\chi_G(3) = 0$? This is an NP-complete problem, so there likely isn't a simple criterion to determine this for a general graph.

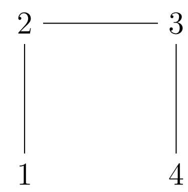
Example 3.5. Let's compute $\chi_G(z)$ for the square:

$G =$ 

It will follow that $\chi(G) = 2$ (or you can figure that out by staring at the square). Here are some different approaches:

(1) For the first way, we just use the definition. If we want to properly k-color G, then 1 can be colored anything, so there are k choices for it. Now the color on 2 and 4 have to be different from the color assigned to 1, so there are $k-1$ choices for each. There are two cases to consider: if the colors of 2 and 4 are the same, then the color for 3 has $k-1$ choices. If they're not the same, then the color for 3 has $k-2$ choices. So the total number of colorings is: $k(k-1)^2 + k(k-1)(k-2)^2$. (The first term counts the number of colorings where 2 and 4 have the same color and the second counts the number of colorings where 2 and 4 have different colors.) We can simplify it to get
$$\chi_G(k) = k(k-1)(k^2 - 3k + 3).$$

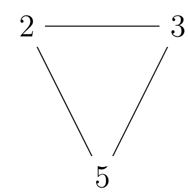
(2) For the second way, we'll use deletion-contraction. Let $e = \{1, 4\}$. Then

$G \setminus e =$ 

Its chromatic polynomial is simple to compute: for a proper k-coloring, 1 has k choices, 2 has $k-1$ choices (any color different from the one given to 1), similarly 3 has $k-1$ choices, and similarly, 4 has $k-1$ choices. So

$$\chi_{G\setminus e}(k) = k(k-1)^3.$$

The contraction by e is

$G/e =$ 

I called the new vertex 5. This is also easy to compute: for a proper k-coloring, 5 has k choices, 2 has $k-1$ choices, and 3 has $k-2$ choices (any color different from the one given to 2 and 5 which are different from each other). So

$$\chi_{G/e}(k) = k(k-1)(k-2).$$

So using Lemma 3.1, we get

$$\begin{aligned}\chi_G(k) &= \chi_{G\setminus e}(k) - \chi_{G/e}(k)\\ &= k(k-1)^3 - k(k-1)(k-2)\\ &= k(k-1)(k^2-3k+3).\end{aligned}$$

(3) A third way (which is sometimes easier but usually harder) is to use polynomial interpolation. That is, we know that $\chi_G(z)$ is a polynomial in z of degree 4, so to determine it, we just need to compute 5 of its values. Some of those are easy: $\chi_G(0) = 0$ (for any G), $\chi_G(1) = 0$ (since G has an edge). Actually, this is already enough to say that $\chi_G(z)$ is divisible by $z(z-1)$. You could also compute by hand that $\chi_G(2) = 2$, $\chi_G(3) = 18$, and $\chi_G(4) = 84$ and then determine the coefficients of $\chi_G(z)$ from this information using linear algebra (though this is a lot of work for this example). □

Example 3.6. Here are some families of graphs where we can give explicit formulas for $\chi_G(z)$ and $\chi(G)$. I won't explain how to get the derivation, you should see if you can figure out how to do it.

(1) The **complete graph** on n vertices is denoted K_n and is defined so that every pair of vertices has an edge between them. Then

$$\begin{aligned}&\chi_{K_n}(z) = z(z-1)(z-2)\cdots(z-n+1),\\ &\chi(K_n) = n.\end{aligned}$$

(2) The cycle C_n of length n has vertices $v_1, \dots, v_n$ and edges $\{i, i+1\}$ for $i = 1, \dots, n-1$ and $\{1, n\}$. Then

$$\chi_{C_n}(z) = \begin{cases} (z-1)^n + (z-1) & \text{if } n \text{ is even} \\ (z-1)^n - (z-1) & \text{if } n \text{ is odd} \end{cases},$$

$$\chi(C_n) = \begin{cases} 2 & \text{if } n \text{ is even} \\ 1 & \text{if } n = 1 \\ 3 & \text{if } n \text{ is odd and } n \geq 3 \end{cases}.$$

(3) If G is a tree with n vertices, then

$$\chi_G(z) = z(z-1)^{n-1}$$

$$\chi(G) = \begin{cases} 1 & \text{if } n = 1 \\ 2 & \text{if } n > 1 \end{cases}. \qquad \square$$