12-fold enumeration
Math 475
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We have k balls and n boxes. We want to count the number of assignments f of balls to boxes. We considered 3 conditions on f: arbitrary (no conditions at all), injective (no box receives more than one ball), surjective (every box has to receive at least one ball). We also considered conditions on the balls: indistinguishable (we can't tell the balls apart) and distinguishable (we can tell the balls apart) and similarly for the boxes: they can be distinguishable or indistinguishable.

balls/boxes	f arbitrary	f injective	f surjective
dist/dist	n^k, see (1)	$(n)_k$, see (2)	$n!S(k,n)$, see (3)
indist/dist	$\binom{n+k-1}{k}$, see (4)	$\binom{n}{k}$, see (5)	$\binom{k-1}{n-1}$, see (6)
dist/indist	$\sum_{i=1}^{n} S(k,i)$, see (7)	$\begin{cases} 1 & \text{if } n \ge k \\ 0 & \text{if } n < k \end{cases}$, see (8)	$S(k,n)$, see (9)
indist/indist	$\sum_{i=1}^{n} p_i(k)$, see (10)	$\begin{cases} 1 & \text{if } n \ge k \\ 0 & \text{if } n < k \end{cases}$, see (11)	$p_n(k)$, see (12)

(1) These are strings of length k in an alphabet of size n.
(2) These are strings of length k without repetitions in an alphabet of size n. Recall that
$$(n)_k = n(n-1)(n-2)\cdots(n-k+1).$$
(3) These are ordered set partitions of $[k]$ into n blocks. Recall that $S(k,n)$ is the Stirling number of the second kind, i.e., the number of partitions of $[k]$ into n blocks.
(4) These are multisets of $[n]$ of size k; equivalently, weak compositions of k into n parts.
(5) These are subsets of $[n]$ of size k.
(6) These are compositions of k into n parts.
(7) These are set partitions of $[k]$ where the number of blocks is $\le n$.
(8) If $n < k$, then we can't assign k balls to n boxes without some box receiving more than one ball (pigeonhole principle), so the answer is 0 in that case. If $n \ge k$, then there is certainly a way to make an assignment, but they're all the same: we can't tell the boxes apart, so it doesn't matter where the balls go.
(9) These are set partitions of $[k]$ into n blocks.
(10) These are the number of integer partitions of k where the number of parts is $\le n$. Remember that $p_i(k)$ is the notation for the number of integer partitions of k into i parts.
(11) The reasoning here is the same as (8).
(12) These are the number of integer partitions of k into n parts.