Finite Coxeter Groups

\[(W, S) \rightarrow \Gamma \text{ Coxeter graph} \]
\[\text{vertex } = S\]
\[\text{between } s \neq t (s \neq t) \text{ draw } m(s, t) \geq 2 \text{ edges.}\]
\[\text{Represent by } \frac{s}{t} \text{ if } m(s, t) = 2\]
\[s \quad \text{if } m(s, t) = 3\]
\[(W, S) \text{ is irreducible if } \Gamma \text{ is connected} \]
\[\text{direct product of } \rightarrow \text{ disjoint union of graphs} \]
\[\text{+ direct sums of } \text{geometric representations} \]

\[G \subset \text{GL}(V) \text{ subgroup acts irreducible on } V \text{ if the only } \]
\[G\text{-invariant subspace of } V \text{ are } 0 \text{ and } V \text{ (} V \text{ is an irreducible representation of } G \text{)} \]

\[g \in \text{GL}(V) \text{ is a reflection if } \cdot g \text{ has finite order} \]
\[\text{im}(g - 1) \text{ is 1-dimensional} \]

Lemma. \[G \subset \text{GL}_n(\mathbb{R}) \text{ act irreducibly on } \mathbb{R}^n. \text{ Assume } G \text{ contains a reflection.} \]
\[= \exists \text{ unique bilinear form (up to scalar multiple) which is } G\text{-invariant} \]
\[(B(v, w) = B'(v', w')) \forall v, w \in \mathbb{R}^n \]
Furthermore, up to sign, this bilinear is symmetric + positive definite.

Proof. If \(B \neq 0\) nonzero \(G\)-invariant bilinear form ,
\[\mathbb{R}^n \rightarrow \left(\mathbb{R}^n\right)^* \text{ is a } G\text{-linear map} \]
\[v \rightarrow B(v, -) \quad \text{By irreducibility, this is an isomorphism.} \]
Let \(B\) be another \(G\)-invariant bilinear form. For any \(v \in \mathbb{R}^n, \)
\[\exists v' \in \mathbb{R}^n \text{ s.t. } B(v, -) = B'(v', -), \text{ if we set } y(v) = v', \]
then \(y : \mathbb{R}^n \rightarrow \mathbb{R}^n \text{ is } G\text{-linear.}\)
Let \( S \in G \) be a reflection, so \( \dim \text{Image}(1-S) = 1 \), let \((1-S)\mathbf{v} \) span it. 
\[ \varphi(1-S)\mathbf{v} = (1-S)\varphi\mathbf{v} = \alpha(1-S)\mathbf{v} \] for some \( \alpha \in \mathbb{R} \).

\( \alpha \) is eigenvalue for \( \varphi \), \( \varphi - \alpha I \) is also \( G \)-linear, its kernel is a nonzero \( G \)-invariant subspace. Irreducibility \( \Rightarrow \ker(\varphi - \alpha I) = \{0\} \)

\[ \Rightarrow B' = \alpha B \]

Consider \( B(v,w) = \sum_{g \in G} g(v) \cdot g(w) \). symmetric, \( G \)-linear

If \( v \neq 0 \), then \( g(v) \cdot g(v) > 0 \) \( \forall g \in G \), \( \Rightarrow B(v,v) > 0 \).

\( \Rightarrow B \) is pos. definite.

(\( \blacksquare \))

Then (Maschke) \( U \) vector space/field of char. 0

\( G \subset GL(U) \) finite subgroup. If \( U' \subset U \) is \( G \)-invariant,

then \( \exists \) \( G \)-invariant subspace \( U'' \) which is complement to \( U' \), i.e.,

\[ U + U'' = U \quad \& \quad U' \cap U'' = 0 \]

(\( \blacksquare \)). Pick any complementary subspace \( X \) to \( U' \). Let \( p: U \rightarrow U' \)

be projection w/ kernel \( X \). Then \( \frac{1}{|G|} \sum_{g \in G} g p \) is also a projection

which is \( G \)-linear. Let \( U'' \) be its kernel.

(\( \blacksquare \)).

Lemma. Let \( (W,S) \) be Coxeter group. Assume \( \Pi \) connected

\& \( W \) is finite. Then \( W \) acts irreducibly on its geometric representation \( V \).

(\( \blacksquare \)). Pick \( V' \subset V \) \( W \)-invariant subspace, define \( S' = \{ s \in S \mid s \cdot V' \} \).

If \( v \in V' \), \( t \notin S' \) then \( s_t(v) - v = -2B(w(v),At)At \in V' \)

\[ \Rightarrow B(w(v),At) = 0 \]. \( \Rightarrow \) no edges between \( S' \) and \( S \setminus S' \) in \( \Pi \).

\( \Pi \) connected \( \Rightarrow \) \( S' = S \) or \( S' = \emptyset \).
Case 1. \( S' = S \Rightarrow V = V' \) ✔

Case 2. \( S' = \emptyset \Rightarrow V' \leq \ker B_w \). We will show \( \ker B_w = 0 \).

Since \( \ker B_w \) is \( W \)-invariant, Marden gives "complement" \( V' \).

If \( V'' = V \), then \( \ker B_w = 0 \) ✔

Otherwise \( V'' \) is a proper subspace, \( \Rightarrow V'' \leq \ker B_w \).

Since \( V'' \cap \ker B_w = 0 \Rightarrow V'' = 0 \Rightarrow \ker B_w = V \implies \quad \square \)

\( \text{Rank} \). Fails if \( W \) infinite: If \( W = \text{infinite dihedral group}, S = \{s, t \} \), we saw that \( \text{span}(d, s + at) \) is a \( W \)-invariant subspace of \( V \).

Prop. \( W \) is finite \( \iff \) \( B_w \) is positive definite.

PF. Assume \( W \) finite, \( \psi \) geometric representation \( W \leq GL(V) \).

Suffices to assume \( \Pi \) is connected, \( \Rightarrow W \subset V \) is irreducible.

\( \Rightarrow B_w \) is either positive definite or negative definite.

\( B_w(a_s, a_s) = 1 \) \( \forall s \in S \Rightarrow \) positive definite.

Suppose \( B_w \) is positive definite. By picking orthonormal basis, group preserving \( B_w \) is \( \{ \psi \mid A^T A = I \} \), which is compact.

Closed & bounded in \( R^{n^2} \)

\( W \leq GL_n(\mathbb{R}) \) is discrete (by previous lecture)

Discrete + compact \( \Rightarrow \) Finite \( \square \)

Thm. Any finite subgroup \( G \leq GL_n(\mathbb{R}) \) generated by reflections is isomorphic to a Coxeter group.

PF. First, \( R^n \) has symmetric positive definite \( G \)-invariant bilinear form

\( (v, w) = \sum g v \cdot g w \). Change basis so that this becomes dot product.
Given hyperplane \( H \subseteq \mathbb{R}^n \), let \( s_H \) be reflection which fixes \( H \) and negates a normal vector. \( H_G = \{ H \mid s_H \in G \} \)

\( \mathbb{R}^n \setminus \cup_{H \in H_G} H \) has finitely many components (called chambers).

Let \( C \) be chamber, \( \overline{C} = \text{closure of } C \).

\( S = \{ s_H \mid H \text{ bounds } \overline{C} \} \). Let \( G' = \text{group generated by } S \).

Claim: Every \( G' \)-orbit of a vector in \( \mathbb{R}^n \) has nonempty intersection with \( \overline{C} \).

Pick \( a \in C \), \( v \in \mathbb{R}^n \). \( G' \cdot v \) is finite \( \Rightarrow \exists \text{ element } v' \in G' \cdot v \) whose distance to \( a \) is minimized. If \( v' \notin \overline{C} \), \( \exists H \text{ bounding } \overline{C} \) s.t. \( a, v' \) are on separate sides of \( H \). \( s_H(v') \) is now closer to \( a \) \( \Rightarrow v' \in \overline{C} \).

Claim \( \Rightarrow G' \) applied to any chamber will contain \( C \)

\( \Rightarrow \) For any \( s_H \in G \), \( \exists g \in G' \) s.t. \( g s_H g^{-1} = s_H \) \( \in S \)

\( \Rightarrow s_H \in G' \Rightarrow G' \) contains all reflections of \( G \) \( \Rightarrow G' = G \).

For each \( s \in S \), let \( a_s \) be unit normal vector to \( \ker(s-1) \) pointing in direction of \( C \)

\( (a_s, a) > 0 \) for all \( a \in C \)

& \( (a_s, a_t) \leq 0 \) for \( s \neq t \). [Reduce to considering \( \text{span}(a_s, a_t) \)]

Claim: \( a_s \) are linearly independent.

Pf: Suppose we have \( \sum_{s \in S} c_s a_s = 0 \). Let \( v = \sum_{s \in S} c_s a_s \).

If \( v = 0 \), then no \( c_s \) are positive. If not, \( v = (a, v) = \sum_{s \in S} c_s (a, a_s) \).
\[ a \in C \quad \Rightarrow \quad 0 = (a, \sum_{s} c_{s} a_{s}) = \sum_{s} c_{s} (a, a_{s}) \quad \forall c_{s} \leq 0, \text{dependency is trivial.} \]

If \( v \neq 0 \), so
\[
0 < (v, v) = \sum_{s \in S} \sum_{t \in S} c_{s} c_{t} (a_{s}, a_{t}) \quad \Rightarrow \quad 0
\]

For \( s, t \in S \), let \( m(s, t) = \text{order of } st \). By restricting to \( \text{span} \{a_{s}, a_{t}\} \), we can conclude that \( (a_{s}, a_{t}) = -\cos \left( \frac{\theta}{m(s, t)} \right) \).

We have surjective homomorphism \( W \to G \) where \( W \) is Coxeter group defined by \( S, m \). \( V \) is geometric representation of \( W \)
\[
\Rightarrow \quad W \to GL(V) \text{ is injective} \quad \Rightarrow \quad W \cong G. \quad \square
\]