A \text{ coves}

\[ D = \{ f \in V^* | f(x) = 0 \text{ for all } s \in S \} \]

\[ E = \{ f \in V^* | f(x) = 1 \} \cong V \]

Let \( \alpha_1, \ldots, \alpha_n \) be simple roots for \( W \) acting on \( V \)

\( \check{\alpha} = \) highest root of \( W \)

**Lemma.** \( D \cap E = \left\{ e_0 + z \mid (z, \alpha_i) \geq 0 \text{ for } i = 1, \ldots, n \right\} \cup \left\{ (z, \check{\alpha}) \leq 1 \right\} \)

**Proof.** Pick \( e_0 + z \in D \cap E \). Then \( (e_0 + z)(x) = 0 \) for all \( x \in S \).

\( (e_0 + z)(s) = 1 \). Note: \( e_0(x) = 0 \) for all \( x \in S \), so \( z(x) \geq 0 \) for all \( x \in S \).

\( e_0(x_0) = 1 \Rightarrow z(x_0) \geq -1 \). When we identify \( (V_a/V_{a_1})^* \cong V \), \( x \in S \) becomes \( \alpha_i \) for some \( i = 1, \ldots, n \) and \( x_0 \) becomes \( -\check{\alpha} \), so \( (z, \alpha_i) \geq 0, i = 1, \ldots, n, \text{ and } (z, \check{\alpha}) \leq 1 \).

Conversely, if these hold, then \( (e_0 + z)(x) = 0 \) for all \( x \in S \) and

\[ 1 = (e_0 + z)(s) = (e_0 + z)(x_0) + z(\delta - x_0) \]

\( (z, \check{\alpha}) \leq 1 \) \( \Rightarrow (e_0 + z)(x_0) = 0 \). \( \square \)

Define \( A_0 = \left\{ v \in V \mid (v, \alpha_i) \geq 0 \text{ for } i = 1, \ldots, n, (v, \check{\alpha}) < 1 \right\} \)

\( A = \left\{ v \in V \mid (v, \alpha_i) \geq 0 \text{ for } i = 1, \ldots, n, (v, \check{\alpha}) \leq 1 \right\} \)

Fundamental alcove (Open simplex), \( A = \overline{A_0} \).

Each \( W_a \)-orbit in \( V \) intersects \( A \) in at most one point.

**Proof.** The \( W_a \)-orbit of \( A \) is \( V \).

**Proof.** Pick \( x \in A_0 \). Pick any \( v \in V \). Note: \( T_v \cdot \mu \) is a
discrete subset of $V$, since $W \uparrow T W \subseteq W$ is finite, $W \backslash \mu$ is also discrete. So $\exists \nu \in W \backslash \mu$ which minimizes distance to $\lambda$. If $\nu \in A$, done. Otherwise, $\exists$ hyperplane that bounds $A$ and separates $\lambda, \nu$. Reflect across hyperplane to get point closer to $\lambda$, but this reflection belongs to $W \mu \rightarrow \nu$.

**Ex.** $A_1$ root system. $\Phi \cong \mathbb{R}$, $\Phi = \{2, -2\}$, so $\alpha_1 = 1$.

$A_\circ = (0, 1) \quad \begin{array}{cccccccc}
2 & 1 & 0 & 1 & 1 & 1 & 3 \\
\end{array}$

$\alpha_1 = e_1 - e_2$
$\alpha_2 = e_2 - e_3$
$\alpha_3 = e_1 - e_3$

$A_2$

$\tilde{B}_2$

$\tilde{C}_2$

**Length Function** we $W \backslash \mu$. $H = H_{x, k}$ separates $A_\circ$ and $W \mu$ if they are on different sides of $H$. 
\[ L(w) = \{ H_{a,k} | H_{a,k} \text{ separates } A_0 \text{ & } wA_0 \} \]

\[ H_{s_{0}} = H_{a,1}, \quad H_{s} = H_{s_{0}} \text{ for } s \in S. \]

**Lemma.** For \( s \in S_a, \quad L(s) = \{ H_{s} \}. \)

**Proof.** Clear that \( H_{s} \in L(s). \) Pick \( x \in A_0. \) Then \( 0 < (x, x) < 1 \)
for all \( a \in \mathbb{R}^+. \) Pick \( t \in S_a \setminus \{ s_{0} \}. \) Then \( (sx, xt) = (x, sa_{t}) \)
and \( sa_t > 0, \) so \( 0 < (sx, xt) < 1 \). \( \Rightarrow \) \( A_0 \) and \( sA_0 \) are on
same side of \( H_{a,k} \) for any \( a \neq a_{s}. \) Since \( (xs, x_{s}) = -(x, a_{s}) \)
and \( 0 < (x, a_{s}) < 1, \) we see that \( H_{a_{s}, k} \) separates only for \( k = 0. \)

**Lemma.** Pick \( w \in w_a, \ s \in S_a. \)

1. \( H_s \) is in exactly one of \( L(w^{-1}) \) or \( L(sw^{-1}) \)
2. \( s(L(w^{-1}) \setminus \{ H_{s} \}) = L(sw^{-1}) \setminus \{ H_{s} \}. \)

**Proof.** (1) If \( x \in w^{-1}A_0, \) then \( (a_{s}, sx) = -(a_{s}, x), \) so \( w^{-1}A_0 \)
and \( sw^{-1}A_0 \) are on different sides of \( H_s. \)

2. Suppose \( H \in L(w^{-1}) \setminus \{ H_{s} \}. \) Since \( SH_s = H_s, \) we know \( S \neq H_s \)
We claim \( sH \in L(sw^{-1}). \) If not, then \( sw^{-1}A_0 \) and \( A_0 \)
are on same side of \( sH. \) \( \Rightarrow \) \( w^{-1}A_0 \) and \( sA_0 \) are same side of \( H. \)
We know \( H \) separates \( w^{-1}A_0 \) and \( A_0. \) \( \Rightarrow \) \( H \) separates \( A_0 \) and \( sA_0. \)
By previous lemma \( \Rightarrow H = H_s. \) \( \Rightarrow \)
\[ s(L(w^{-1}) \setminus \{ H_{s} \}) \subseteq L(sw^{-1}) \setminus \{ H_{s} \} \]
Reverse inclusion follows by symmetry.
Prop. Pick reduced expression \( w = s_{i_1} \ldots s_{i_r} \) for \( w \in \mathcal{W}_a \).

(1) The hyperplanes \( H_{s_{i_1}}, s_{i_1}H_{s_{i_2}}, \ldots, s_{i_1} \ldots s_{i_{r-1}} H_{s_{i_r}} \) are distinct.

(2) \[ \mathcal{L}(w) = \phi \]

Proof. (1) If \( n \not\equiv 0 \mod 4 \), then \( s_{i_1} \ldots s_{i_{r-1}} H_{s_{i_r}} = s_{i_1} \ldots s_{i_{r-1}} H_{s_{i_r}} \)

\[ \implies H_{s_{i_1} \ldots s_{i_{r-1}}} = H_{s_{i_1} \ldots s_{i_r}}. \]

From last lecture, we know \( \exists x \in L_{w_0} \) and \( u \in W_{s_{i_1} \ldots s_{i_{r-1}}} = L_w \).

\[ \implies (s_{i_1} \ldots s_{i_{r-1}})(s_{i_1} \ldots s_{i_{r-1}})^{-1} = s_{i_1}. \]

\[ \implies s_{i_1} \ldots s_{i_{r-1}} s_{i_r} = s_{i_1} \ldots s_{i_{r-1}}. \]

(2) Induction on \( l(w) \). \( l(w) = 0 \) clear.

Assume \( l(w) > 0 \). By induction

\[ \mathcal{L}(s_{i_1} w) = \mathcal{L}(H_{s_{i_2}}, s_{i_2} H_{s_{i_3}}, \ldots, s_{i_2} \ldots s_{i_{r-1}} H_{s_{i_r}}) \]

and by (1), set has size \( r-1 \). Apply \( s_{i_1} \) to all to get another set of \( r-1 \) hyperplanes, not containing \( H_{s_{i_1}} \) by (1).

\[ \implies \mathcal{L}(s_{i_1} w) \neq H_{s_{i_1}}. \]

By previous lemma, \( H_{s_{i_1}} \in \mathcal{L}(w) \) and

\[ s_{i_1} (\mathcal{L}(w) \setminus \{H_{s_{i_1}}\}) = \mathcal{L}(s_{i_1} w). \]

Cor. For \( w \in \mathcal{W}_a \), \( l(w) = |\mathcal{L}(w)| \).