Affine representation of affine Weyl groups

\((w_n, s_a) = \text{Coxeter group whose graph is connected and positive semidefinite (but not pos.def.)} \)

\(V_a = \text{geometric representation, } n+1 = \text{rank } (w_n, s_a) \)

\(X_n = \text{name from classification} \)

\(s_0 \in s_a \text{ s.t. } (w, s_a \backslash s_0) \text{ is of type } X_n \)

\(\ker B_w \text{ is } 1\text{-dim, spanned by } s \text{ s.t. coeff. of } \alpha_0 \text{ is } 1 \)

\(V_a^+ \)

Define \(Z = \{ f \in V_a^* \mid f(s) = 0 \} \)

\(E = \{ f \in V_a^* \mid f(s) = 1 \} \)

\(Z \text{ linear space, } E \text{ affine space over } Z \)

\(\text{i.e., have simply transitive action } \mathbb{Z} \times E \to E \)

**Def.** \(\Psi: E \to E \text{ is an affine transformation if there exists a linear map } \Psi: Z \to Z \text{ s.t. } \Psi(e+z) = \Psi(e) + \Psi(z) \ \forall e \in E, z \in \mathbb{Z} \)

\(\text{Aff}(E) = \text{group of affine transformations under composition.} \)

**Lemma.** \(\text{Aff}(E) \cong \{ g \in GL(V_a^*) \mid g(E) = E \} \)

**If.** Pick \(g \in GL(V_a^*) \text{ s.t. } g(E) = E \). For \(e \in E, z \in \mathbb{Z}, \)
we have \(g(e+z) = g(e) + g(z) \), so \(g: E \to E \text{ is affine.} \)

Conversely, suppose \(\Psi: E \to E \text{ is affine, let } \Psi: Z \to Z \text{ be corresponding linear map. Pick basis } V_0, \ldots, V_n, \text{ s.t. } v_0 = 1, \text{ and } \)

...
for $V_A$. Let $v_0^*, \ldots, v_n^*$ be dual basis for $V_{A^*}$.
Then $v_1^*, \ldots, v_n^*$ is basis for $E$ and we define $g \in GL(V_{A^*})$
by $g(c_0 v_0^* + \cdots + c_n v^n_*) = c_0 \Psi(v_0^*) + \Psi(c_1 v_1^* + \cdots + c_n v^n_*)$.

$$g((c_0 + c') v_0^* + \cdots + (c_n + c'n') v^n_*) = c_0 \Psi(v_0^*) + \Psi(c_1 v_1^* + \cdots + c_n v^n_*)$$
$$+ c'_0 \Psi(v_0^*) + \Psi(c_1' v_1^* + \cdots + c'n v^n_*)$$

$\Rightarrow$ linear

$g(E) = E$ ( $E$ corresponds to $c_0 = 1$) and $g|E = \Psi$ since any element of $E$ is of the form $v_0^* + v'$ where $v' \in \text{span} v_1^*, \ldots, v_n^*$, and so $g|E(v_0^* + v') = \Psi(v_0^*) + \Psi(v') = \Psi(v_0^* + v')$. 

$\square$

\textbf{Note.}$ w 8 = 8 \ A \text{ we } W_A \text{, so } W_A \text{ preserves both } Z, E$

$\Rightarrow \text{ homomorphism } W_A \rightarrow \text{ Aff } (E)$

This is injective.

$B_{W_A}$ descends to pos. def. form on $V_A / V_A^+$ (call it $B_w$)
and we have $Z \cong (V_A / V_A^+)^*$. Using $B_w$, we can identify $(V_A / V_A^+) \cong (V_A / V_A^+)^*$

$$v_f \rightarrow f$$

where $v_f$ is unique vector s.t. $f(x) = B_w(v_f, x) \forall x \in V_A / V_A^+$.

In particular, get $W_A$-invariant pos. def. form on $Z$.

For $se S_A$, define

$$Z_s = \{ f \in V_A^* | f(s) = 0 \}$$

$$E_s = E \cap Z_s$$
Since coeff of $a_0$ in $S$ is 1, $\{x_5 | s \in S \setminus \{0\}\}$ is linearly independent, so $E \cap \bigcap_{s \in S} \mathbb{Z}_s$ is a single point $e_0$. \\

Explicitly, $e_0(S) = 1$, $e_0(x_5) = 0$ for $S \neq 0$.

$\downarrow$

$e_0(x_0) = 1$

$\Rightarrow$ Identify $\mathbb{Z} \cong E$ via $z \mapsto z + e_0$.

Inverse denoted by $e' \mapsto e' - e_0$ will form $BE$ on $E$

**Lemma.** $BE$ is invariant under $(W_a)e_0$.

**Pf.** Pick $w \in (W_a)e_0$, so $w e_0 = e_0$. Pick $e', e'' \in E$.

$BE(w e', w e'') = BW(w e' - e_0, w e'' - e_0)$

$= BW(w(e' - e_0), w(e'' - e_0))$

$= BW(e' - e_0, e'' - e_0)$

$= BE(e', e'')$. $\square$

**Prop.** $W_a$ is isomorphic to subgroup of Aff(E) which is generated by affine reflections.

**Pf.** Pick $s \in S$. $E_s$ is an affine hyperplane passing through $e_0$. $S$ fixes $E_s$, so fixes $e_0$, so preserves $BE$.

Since $s^2 = 1$, it must be reflection w.r.t. $E_s$.

Now consider $s_0$. Let $\alpha \in \mathbb{Z}$ be the linear functional $v \mapsto BW(a_s, v)$. Given $v \in V$, $z \in \mathbb{Z}$,
\((S_0(e_0 + z))(v) = (e_0 + z)(S_0(v)) \)

\[= (e_0 + z)(v - 2B_{W_\alpha}(v, \alpha_{S_0}) \alpha_{S_0}) \]

\[= (e_0 + z)(v) - 2(e_0 + z)(\alpha_{S_0}) \alpha'_0(v) \]

\[= (e_0 + z)(v) - 2(1 + B_{W}(z, \alpha'_0)) \alpha'_0(v) \]

\[\Rightarrow S_0(e_0 + z) = (e_0 + z) - 2(1 + B_{W}(z, \alpha'_0)) \alpha'_0 \]

\[\Rightarrow S_0 \text{ is an affine reflection} \quad \square\]