Def. Length function \( l \) on \( W \) is defined by \( l(w) = \text{minimum } n \text{ s.t. } w = s_1 s_2 \ldots s_n \) where \( s_i \in S \). A reduced expression for \( w \) is any expression \( w = s_1 \ldots s_n \) where \( n = l(w) \). \( l(1) = 0 \) by convention.

Lemma. For all \( w, w' \in W \), have:
1. \( l(w) = l(w^{-1}) \)
2. \( l(w) = 1 \iff w \in S \)
3. \( l(ww') \leq l(w) + l(w') \)
4. \( l(ww') \geq l(w) - l(w') \)
5. For all \( s \in S \), \( l(ws), l(sw) \in \mathbb{Z} \), \( l(w) = l(w^{-1}) \).

Proof:
1. If \( w = s_1 \ldots s_n \) is reduced, so is \( w^{-1} = s_n^{-1} \ldots s_1^{-1} = s_n \ldots s_1 \).
2. Clear.
3. \( w = s_1 \ldots s_n \) reduced, \( w' = s_1' \ldots s_n' \) reduced,
   \( ww' = s_1 \ldots s_n s_1' \ldots s_n' \Rightarrow n + m \geq l(ww') \).
4. By 3, \( l(ww') + l(w^{-1}) = l(w) \), substitute \( l(w') \).
   \[ l(ww') + l(w') \]
5. \( l(ws) \leq l(w) + l(s) = l(w) + 1 \) by 3.
   \[ l(ws) \geq l(w) - l(s) = l(w) - 1 \] by 4.

Note: if \( l(w) = l(ws) \), then \( \text{sgn}(w) = (-1)^{l(w)} = (-1)^{l(ws)} = \text{sgn}(ws) = -\text{sgn}(w) \).

Define root system of \( W \) to be \( \{ w(\alpha_s) \mid s \in S, w \in W \} \).

The elements are called roots. A root \( \alpha \) is positive \((\alpha > 0)\) if \( \alpha \) is a non-negative linear combination of \( \{ \alpha_s \mid s \in S \} \). A root \( \alpha \) is negative \((\alpha < 0)\) if \( \alpha \) is a non-positive linear combination of \( \{ \alpha_s \mid s \in S \} \).
Ex. \( W \) is finite dihedral group, 
\[ S = \{ s, t^j \}, \quad m(s, t) = m < \infty \]

The geometric representation of \( W \) is isometric to \( \mathbb{R}^2 \) with standard inner product
so that \( x_s = (1, 0), \quad x_t = (\cos \theta, \sin \theta), \quad t = \frac{\pi}{m}. \)

The roots are points \((\cos(\frac{j\pi}{m}), \sin(\frac{j\pi}{m}))\), \(0 \leq j < 2m, \)
vertices of regular \(2m\)-gon.

\( t \cdot s = \) rotation by \( \frac{2\pi}{m} \)

- positive roots
- negative roots (all \(-1\) times a positive root)

\[ \Rightarrow \text{Every root is either positive or negative.} \]

Given subset \( I \subseteq S \), define \( W_I \subseteq W \) subgroup generated by \( I \).

\( l_I : W_I \rightarrow \mathbb{Z}_{\geq 0} \) defined by \( l_I(w) = \text{minimal } n \text{ s.t. } w = s_1 \cdots s_n, \ s_i \in I \).

Thm. Pick \( w, s \in S \).

1. If \( l(ws) > l(w) \), then \( w(s) > 0 \) (positive root).
2. If \( l(ws) < l(w) \), then \( w(s) \) is a negative root.

pf. 2 follows from 1 by using \( ws \) in place of \( w \):
if \( l(ws) < l(w) \), then \( l_I(ws) > l(w) \)
\[ \Rightarrow \quad w(s) > 0 \quad \Rightarrow \quad w(-s) < 0. \]

We prove 1 by induction on \( l(w) \). If \( l(w) = 0 \), then \( w = 1 \), so \( l(s) > 0 \) \( \forall s \in S \)
& \( l(1) = \alpha_s > 0. \)

Now assume \( l(w) > 0 \), pick \( t \in S \) s.t. \( l(wt) < l(w) \). So \( t \neq 5 \), set \( I = \{ s, t \} \).

Define \( A = \{ (x, x_I) \in W \times W^+ \mid w = xx_I, \ l(w) = l(x) + l_I(x_I) \} \).

Note: \((wt, t) \in A \), so \( A \neq \emptyset \). Pick \((v, v_I) \in A \) s.t. \( l(v) \) is minimal possible.

Then \( l(v) \leq l(wt) = l(w) - 1 \).

Claim: \( l(us) > l(v) \) \& \( l(vt) > l(v) \)

Suppose \( \neg \). Then \( l(w) \leq l(us) - 1 + l(sv_I) = (l(v) - 1) + l(sv_I) \).
\[ \leq (l(v) - 1) + l_I(sv_I) \]
\[ \leq (l(v) - 1) + 1 + l_I(v_I) = l(v) + l_I(v_I) = l(w) \]
\( \Rightarrow \) All \( \leq \) are equalities. \( \Rightarrow \) \( l(w) = l(vs) + l(sv_t) \)
\( \Rightarrow (vs, sv_t) \in A \). Since \( l(vs) < l(v) \), this contradicts choice of \( v \). \( \square \)

By induction, \( v(d_s) > 0 \), \( v(d_t) > 0 \). Sufﬁces to prove \( v_I(d_s) > 0 \) since \( v_I(d_s) \in \text{Span} \{ d_s, d_t \} \) \& \( w = vv_I \).

A reduced expression for \( v_I \) using \( s_i \) is an alternating product of \( s, t \). This must end in \( t \) (if not, then since \( l(w) = l(v) + l_I(v_I) \), and \( w = vv_I \), we get \( l(ws) < l(w) \) \( \Rightarrow \)).

Let \( m = n(s, t) \). Case 1: \( m < \infty \). By explicit example.
Case 2. \( m = \infty \). \( v_I = (st)^k \) or \( t(st)^k \), \( k \geq 0 \).

\( (s, st)^k(d_s) = (2k+1)d_s - 2k a_t > 0 \)

\( s_t(s, st)^k(d_s) = s_t(2k+1)d_s + 2k a_t = (2k+1)d_s + 2(k+2)a_t > 0 \). \( \square \)

Cor. Every root is either positive or negative.

\[ \mathfrak{b} = \mathfrak{b}^+ \cup \mathfrak{b}^- \], \( \mathfrak{b}^- = -\mathfrak{b}^+ \).

\( \forall w \in W, s \in S \)

Cor. \( \sigma : W \rightarrow GL(V) \) is injective:
\( \forall w \neq 1 \). \( \exists s \in S \) \( \sigma(w) = s \).

By Thm \( a_s = w(a_s) < 0 \), contradiction. \( \square \)

Ex. \( \alpha_{n-1} \) Coxeter system \( W \rightarrow G_n = \text{image of } \sigma \)
\( \Rightarrow \) \( W \cong G_n \)
\( \Rightarrow \) \( G_n \cong \langle s_i, \ldots, s_{n-1} \mid s_i^2 = 1 \ \forall i \quad (s_i s_j)^2 = 1 \text{ for } |i-j| > 1 \rangle \quad (s_i s_{i+1})^3 = 1 \ \forall i = 1, \ldots, n-2 \)