Jacobian

Setup: $h_1, \ldots, h_n \in \mathbb{C}[x_1, \ldots, x_j]$

Jacobian: $J = J(h_1, \ldots, h_n) = \det \left( \frac{\partial h_i}{\partial h_j} \right)_{i, j = 1, \ldots, n}$

Lemma: $h_1, \ldots, h_n$ are algebraically independent $\iff$ $J \neq 0$.

Proof: Suppose $h_1, \ldots, h_n$ are dependent. \exists non-zero polynomial $F(y_1, \ldots, y_n)$ sat. $F(h_1, \ldots, h_n) = 0$. Pick $F$ w/ smallest possible degree. Apply $\frac{\partial}{\partial x_j}$:

$$0 = \frac{\partial}{\partial x_j} F(h_1, \ldots, h_n) = \sum_{i=1}^n \frac{\partial F}{\partial y_i}(h_1, \ldots, h_n) \frac{\partial h_i}{\partial x_j}$$

In matrix form:

$$\left[ \frac{\partial F}{\partial y_1}(h_1, \ldots, h_n) \ldots \frac{\partial F}{\partial y_n}(h_1, \ldots, h_n) \right] \left[ \frac{\partial h_1}{\partial x_j} \ldots \frac{\partial h_n}{\partial x_j} \right]_{i, j = 1, \ldots, n} = 0$$

$F$ is not constant $\Rightarrow$ some $\frac{\partial F}{\partial y_j} \neq 0$. By minimality of $\deg F$,

$$\frac{\partial F}{\partial y_j}(h_1, \ldots, h_n) \neq 0 \Rightarrow J \det \left( \frac{\partial h_i}{\partial x_j} \right) = 0.$$

Now suppose $h_1, \ldots, h_n$ are algebraically independent. For each $i$, $\exists x_i, h_1, \ldots, h_n$ is algebraically dependent. For each $i$, let $F_i$ be a polynomial in $y_0, y_1, \ldots, y_n$ of minimal possible degree sat. $F_i(x_i, h_1, \ldots, h_n) = 0$. Apply $\frac{\partial}{\partial x_j}$:

$$0 = \frac{\partial}{\partial x_j} F_i(x_i, h_1, \ldots, h_n) = \frac{\partial F_i}{\partial y_0}(x_i, h_1, \ldots, h_n) \delta_{ij} + \sum_{k=1}^n \frac{\partial F_i}{\partial y_k}(x_i, h_1, \ldots, h_n) \frac{\partial h_k}{\partial x_j}$$
In matrix form:

\[
\left( \frac{\partial F_i}{\partial y_j} (x_i, h_1, \ldots, h_n) \right)_{i,j=1,\ldots,n} = - \left( \frac{\partial F_i}{\partial y_0} (x_i, h_1, \ldots, h_n) S_{ij} \right)_{i,j=1,\ldots,n}
\]

Since \( h_1, \ldots, h_n \) are alg. ind., \( F_i \) is positive degree wrt \( y_0 \)

\[ \Rightarrow \frac{\partial F_i}{\partial y_0} \neq 0 \text{ for all } i. \Rightarrow \frac{\partial F_i}{\partial y_0} (x_i, h_1, \ldots, h_n) \neq 0 \text{ by minimality of } \deg F_i \]

\[ \text{diagonal matrix w/ nonzero entries } \Rightarrow \det \neq 0 \]

\[ \Rightarrow J \neq 0 \]

As before, \( W \subset \text{GL}_{n \mathbb{C}} \) complex reflection group.

\( f_1, \ldots, f_n \in \mathbb{C}[x_1, \ldots, x_n] \) generators for \( \mathbb{C}[x_1, \ldots, x_n]^W \).

Chevalley \( \Rightarrow f_1, \ldots, f_n \) alg. ind. \( \Rightarrow J = J(f_1, \ldots, f_n) \neq 0. \)

\( \deg J = \sum_{i=1}^n (\deg f_i - 1) \)

\[ \text{Ex. } W = G_n \subset \text{GL}_{n \mathbb{C}}. \quad f_i = \frac{1}{i} (x_1^i + \ldots + x_n^i) \]

\[ J = \det \left( \frac{\partial f_i}{\partial x_j} \right) = \det \left( x_j^{i-1} \right) = \begin{pmatrix}
1 & 1 & \cdots & 1 \\
x_1 & x_2 & \cdots & x_n \\
x_1^2 & x_2^2 & \cdots & x_n^2 \\
\vdots & \vdots & \ddots & \vdots \\
x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1}
\end{pmatrix}
\]

\[ = \prod_{1 \leq i < j \leq n} (x_j - x_i) \neq 0. \]