Ring of invariants

\[ G = \text{finite group}, \quad G \subseteq A = \text{Sym}(V), \quad V \text{ complex vector space} \]

\[ \mathcal{P} = \frac{1}{|G|} \sum_{g \in G} g, \quad \text{Define } f^\# = \mathcal{P}(f) \text{ for } f \in A \]

\[ \text{Reynolds operator} \]

If \( f_1, f_2 \in A^G \), then \( (f_1 f_2)^\# = f_1 f_2^\# \)

\[ \mathcal{P}(f_1 f_2) = \frac{1}{|G|} \sum_{g \in G} g(f_1) g(f_2) = \frac{f_1}{|G|} \sum_{g \in G} g(f_2) = f_1 f_2^\# \]

\[ \Rightarrow \quad \# : A \rightarrow A^G \text{ is a } A^G \text{-module homomorphism.} \]

\[ + \text{ surjective + preserves degrees} \]

Prop. Let \( I \subseteq A \) be ideal generated by \( A^G \):

\[ I = \{ \sum \alpha_i f_i \mid f_i \in A^G, \alpha_i \in A \} \]

Suppose that \( f_1, \ldots, f_k \in A^G \) are positive degree homogeneous elements that generate \( I \). Then \( f_1, \ldots, f_k \) generate \( A^G \) as a \( \mathbb{C} \)-algebra.

Pf. Pick \( f \in A^G \). We show \( f \) is generated by \( f_1, \ldots, f_k \) by induction on \( \text{deg}(f) \).

\[ f \in A^G \Rightarrow f \in I \Rightarrow \exists h_1, \ldots, h_k \in A \text{ homogeneous s.t. } f = h_1 f_1 + \ldots + h_k f_k. \]

Apply \( \# \):

\[ f = f^\# = h_1^\# f_1 + \ldots + h_k^\# f_k. \]

Note \( \text{deg} h_i^\# + \text{deg} f_i = \text{deg} f \). By induction, \( h_i^\# \) is generated by \( f_1, \ldots, f_k \) as a \( \mathbb{C} \)-algebra. Substitute these expressions in for \( h_i^\# \) to get that \( f \) is generated by \( f_1, \ldots, f_k \) as \( \mathbb{C} \)-algebra. \( \square \)
Cor. \( AG \) is a finitely generated \( F \)-algebra.

Prf. \( I \) is f.g. by Hilbert basis thm. \( \square \)

Rmk. Proof extends to any field of characteristic 0.

In general, \( AG \) is f.g. (Noether) Let \( k \) be any field.

Let \( t \) be a new variable. For each \( i \), consider
\[
P_i(t) = \prod_{g \in G}(t - g x_i) \in A[t].
\]
In fact, \( p_i(t) \in AG[t] \).

Let \( B = \{ k \text{-subalgebra of } AG \text{ generated by the coeff. of } p_i(t) \text{ for all } i \} \). By definition, \( B \) is finitely generated over \( k \).

Furthermore, \( A \) is a finitely generated \( B \)-module:
\[
\{ x^j_i \mid i = 1, \ldots, n; 0 \leq j < |G| \} \text{ generates } A \text{ as a } B \text{-module:}
\]
\[
x_i^j = \text{ linear combination of lower powers of } x_i \text{ w/ coeff. in } B.
\]
Since \( p_i(x_i) = 0 \).

Next, \( AG \) is a \( B \)-submodule of \( A \). Hilbert basis thm \( \Rightarrow \)

\( AG \) is a f.g. \( B \)-module. A set of generators as \( B \)-module

together w/ generators for \( B \) give set of algebra generators for \( AG \). \( \square \)

Def. \( R \) integral domain, \( \text{Frac}(R) = \text{field of fractions} \)

Prop \( \text{Frac}(AG) = \text{Frac}((A)^G) \).

In particular, if \( G \) acts faithfully on \( V \), then \( \text{Frac}(H) \) is a degree \( |G| \) extension of \( \text{Frac}(A^G) \), and tr. deg. \( \text{Frac}(A^G) = n \).
\begin{proof}
\text{Frac}(A^G) \subseteq \text{Frac}(A)^G$. Pick \( \frac{p}{q} \in \text{Frac}(A)^G \).
Define \( q' = \prod_{g \in G} g \cdot p \). Then \( pp' \) is \( G \)-invariant. \( \frac{p}{q} = \frac{pp'}{q'} \).
\( q'p' \) is also \( G \)-invariant \( \Rightarrow \frac{p}{q} \in \text{Frac}(A^G) \).

In general, if \( G \subseteq \text{Aut}(K) \), then \( K \) is a degree \( |G| \) extension of \( KG \).

Since \( \text{tr.deg} \) is constant within finite extensions,
\[ \text{tr.deg}(\text{Frac}(A^G)) = \text{tr.deg}(\text{Frac} A) = n. \]
\end{proof}