Polynomial Invariants

If \( k \) is a field, \( V \) is a vector space of dimension \( n \), pick a basis \( x_1, \ldots, x_n \) for \( V \).

\[
A = k[x_1, \ldots, x_n] = \text{Sym}(V) = \bigoplus_{d \geq 0} \text{Sym}^d(V)
\]

\( G \) is a finite group acting on \( V \), \( G \) also acts on each \( \text{Sym}^d(V) \), and \( A \).

Since \( (V^*)^* = V \), \( A \) is ring of polynomial functions on \( V^* \).

An ideal \( I \subset A \) (nonempty) s.t. if \( f, g \in I \) then \( f + g \in I \) and if \( f \in I, \text{deg}(f) = d \), then \( f \in I \). A set \( S \) generates \( I \) if \( I = \bigoplus_{s \in S} s \) \text{fin-gens}(A) \text{fin-gens}(A)

A finitely generated \( k \)-algebra is a quotient ring of \( A \) by an ideal.

Thus (Hilbert basis theorem). Let \( R \) be a finitely generated \( k \)-algebra.

Every ideal of \( R \) has a finite generating set.

More generally, if \( M \) is a finitely generated \( R \)-module, then every submodule of \( M \) is finitely generated.

Grading. Pick positive integers \( d_1, \ldots, d_n \), set \( \text{deg}(x_i) = d_i \). \( X_1^{d_1} \ldots X_n^{d_n} \) s.t. \( d_1 p_1 + \ldots + d_n p_n = d \). (Note: \( A_0 = k \) spanned by constant poly)

A module \( M \) is graded if \( M = \bigoplus_{d \geq 0} M_d \) such \( \forall f \in A_d, meM_e, fmeM_{d+e} \).

An element of \( M_d \) is called homogeneous.

An ideal \( I \) is homogeneous if \( I = \bigoplus_{d \geq 0} (I \cap A_d) \).

In this case, \( I \) is graded w/ \( I_d = I \cap A_d \).

Homogeneous modules always have generating sets consisting of homogeneous elements.
If \( M \) is graded module, its Hilbert series is
\[
H_M(t) = \sum_{d \geq 0} (\dim_k M_d) t^d.
\]

Given graded modules \( M, N \), their tensor product is graded via
\[
(M \otimes N)_d = \bigoplus_{e=0}^d M_e \otimes N_{d-e}.
\]

\[
H_{M \otimes N}(t) = H_M(t) H_N(t).
\]

Important case: \( k[x_1, \ldots, x_n] = k[x_1] \otimes_k \ldots \otimes_k k[x_n] \)
\[
H_{k[x_1, \ldots, x_n]}(t) = \prod_{i=1}^n \frac{1}{1 - t \deg(x_i)}.
\]

F_1, \ldots, F_k \in A are algebraically independent if, for any nonzero polynomial \( h(y_1, \ldots, y_k) \) (\( y_i \)'s are new variables), we have \( h(f_1, \ldots, f_k) \neq 0 \).

Any alg. ind. set becomes alg. ind. set in \( \text{Frac}(k[x_1, \ldots, x_n]) = k(x_1, \ldots, x_n) \)
which can be extended to a transcendence basis over \( k \)
\Rightarrow alg. ind. sets have size \( \leq n \)

Molien's formula \( \text{Let } k = \mathbb{C} \)

**Lemma.** \( V = \) finite dim. vector space, \( W = \) finite group \( G \) acting on it.

Define \( \Psi : V \to V \) by \( \Psi(g) = \frac{1}{|G|} \sum_{g \in G} g \cdot v \).

Then

1. \( \Psi \) is a projection, i.e., \( \Psi^2 = \Psi \)
2. The image of \( \Psi \) is \( V^G \)
3. \( \dim V^G = \text{Tr}(\Psi) = \frac{1}{|G|} \sum_{g \in G} \text{trace}(g | V) \)
First, show \( \mathbb{im}(\varphi) = V^G \), i.e., \( \varphi(g) = \varphi(v) \) for all \( g \in G \).

Pick \( h \in G \), \( v \in V \). Then
\[
h \cdot \varphi(v) = \frac{1}{|G|} \sum_{g \in G} h(g \cdot v) = \frac{1}{|G|} \sum_{g \in G} g \cdot v = \varphi(v).
\]

If \( v \in V^G \), \( \varphi(v) = \frac{1}{|G|} \sum_{g \in G} w = w \), so \( \mathbb{im}(\varphi) = V^G \). \( \Rightarrow \) \( \varphi^2 = \varphi \) b/c \( \forall v \in V \), \( \varphi^2(v) = \varphi(\varphi(v)) = \varphi(v) \). \( \Rightarrow \) \( \varphi^2 = \varphi \) b/c \( \forall v \in V^G \).

For any projection, its eigenvalues are either 0 or 1;
its rank is multiplicity of 1, which is its trace.

\( \Rightarrow \) \( \text{trace } \varphi = \text{dim } V^G \).

\( \text{Thm (Molien's formula). Let } G \text{ act on } V, \ A = \text{Sym}(V) \)

\( \text{let } \varphi(g) \text{ be linear operator of } g \text{ acting on } V. \text{ Then} \)
\[
\sum_{d \geq 0} \dim (\text{Sym}^d V)^G t^d = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(1 - \varphi(g)t)}
\]

Pick \( g \in G \). Let \( z_1, \ldots, z_n \) be eigenvalues (w/ multiplicity) of \( \varphi(g) \).

Let \( V_1, \ldots, V_n \) be eigenbasis for \( \varphi(g) \). An eigenbasis for \( g \) acting on \( \text{Sym}^d V \) is \( \{ V_{i_1} \cdots V_{i_d} \mid 1 \leq i_1 \leq \cdots \leq i_d \leq n \} \).

\( \sum_{d \geq 0} \text{Tr}(g|\text{Sym}^d V) t^d = \frac{1}{z_1 \cdots z_n} = \frac{1}{\det(1 - \varphi(g)t)}
\]

\( \Rightarrow \) \( \sum_{d \geq 0} \dim (\text{Sym}^d V)^G t^d = \frac{1}{|G|} \sum_{g \in G} \text{Tr}(g|\text{Sym}^d V) t^d = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(1 - \varphi(g)t)} \) \( \Box \)