Math 207A, Fall 2023
Homework 1 (corrected on October 20, 2023)

Here are some problems related to our discussion in class. In some sense, they are miscellaneous calculations, but working through them carefully will give you a better handle on the representation theory of semisimple Lie algebras.

If you're doing this for credit, you can skip the optional problems, but they provide some extra context.

1. Spherical harmonics

Recall the notation: our representation is $P = \mathbf{C}[z_1, \dots, z_n]$ and we defined operators on P:

$$q = \sum_{i=1}^n z_i^2, \qquad E = \sum_{i=1}^n z_i \partial_i, \qquad \Delta = \sum_{i=1}^n \partial_i^2.$$

We defined a representation $\rho\colon \mathfrak{sl}_2 \to \mathfrak{gl}(P)$ by

$$\rho(X) = \frac{1}{2}\Delta, \qquad \rho(H) = -E - \frac{n}{2}I, \qquad \rho(Y) = -\frac{1}{2}q.$$

(1) For each $d \geq 0$, let P_d be the space of homogeneous degree d polynomials and let $qP_d \subset P_{d+2}$ denote the image of multiplication of P_d by q (i.e., the degree $d+2$ subspace of the ideal generated by q).

Let $P'_{d+2} \subset P_{d+2}$ denote the space of harmonic polynomials, i.e.,

$$P'_{d+2} = \{f \in P_{d+2} \mid \Delta f = 0\}.$$

Construct a basis for P'_{d+2} and show that we have a direct sum decomposition

$$P_{d+2} = qP_d \oplus P'_{d+2}.$$

We will define $P'_0 = P_0$ and $P'_1 = P_1$.

(2) (Optional) Show that P'_d is an irreducible $\mathbf{O}_n(\mathbf{C})$-subrepresentation of P_d.

(3) Show that the $\mathfrak{sl}_2$-subrepresentation V_d generated by P'_d has the following description:

$$V_d = \{q^r f \mid r \geq 0,\ f \in P'_d\}.$$

(4) Show that we have a direct sum decomposition of $\mathfrak{sl}_2$-representations

$$P \cong \bigoplus_{d \geq 0} V_d.$$

(5) Describe V_d in terms of Verma modules.

2. Classical Lie algebras

(6) The Killing form κ is a nondegenerate symmetric bilinear form, so for a semisimple Lie algebra $\mathfrak{g}$, the image of the adjoint representation ad is contained in $\mathfrak{so}(\mathfrak{g}, \kappa)$. Show that when $\mathfrak{g} = \mathfrak{sl}_2$, this gives an isomorphism $\mathrm{ad}\colon \mathfrak{sl}_2 \to \mathfrak{so}(\mathfrak{sl}_2, \kappa) \cong \mathfrak{so}_3$.

(7) Construct a symplectic form ω on $\mathbf{C}^2$ which is stabilized by $\mathfrak{sl}_2$ and use this to construct an isomorphism $\mathfrak{sl}_2 \cong \mathfrak{sp}_2$.

(8) Using the notation from the previous problem, define β on $\mathbf{C}^2 \otimes \mathbf{C}^2$ by

$$\beta(\sum_i x_i \otimes y_i, \sum_j x'_j \otimes y'_j) = \sum_{i,j} \omega(x_i, x'_j)\omega(y_i, y'_j).$$

Show that β is symmetric and nondegenerate. Furthermore, define a representation of $\mathfrak{sl}_2 \times \mathfrak{sl}_2$ on $\mathbf{C}^2 \otimes \mathbf{C}^2$ by

$$(A, B) \sum_i (x_i \otimes y_i) = \sum_i (Ax_i \otimes y_i + x_i \otimes By_i).$$

Show that $\mathfrak{sl}_2 \times \mathfrak{sl}_2$ stabilizes β and use this to construct an isomorphism between $\mathfrak{sl}_2 \times \mathfrak{sl}_2$ and $\mathfrak{so}(\mathbf{C}^2 \otimes \mathbf{C}^2, \beta) \cong \mathfrak{so}_4$.

(9) Given a representation V of $\mathfrak{g}$, we have defined a representation of $\mathfrak{g}$ on $V^{\otimes k}$. Show that the subspace of skew-symmetric tensors, denoted $\bigwedge^k V$, is a subrepresentation. As usual, for $v_1, \dots, v_k \in V$, we define

$$v_1 \wedge \cdots \wedge v_k = \sum_\sigma \operatorname{sgn}(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}$$

where the sum is over all permutations on k letters.

Now consider the case $\mathfrak{g} = \mathfrak{sl}_4$ and $V = \mathbf{C}^4$ is the space of column vectors. Consider the usual multiplication map

$$\beta \colon \bigwedge^2 \mathbf{C}^4 \otimes \bigwedge^2 \mathbf{C}^4 \to \bigwedge^4 \mathbf{C}^4$$

which is defined on simple tensors by $(v_1 \wedge v_2) \otimes (v_3 \wedge v_4) \mapsto v_1 \wedge v_2 \wedge v_3 \wedge v_4$. Since $\bigwedge^4 \mathbf{C}^4$, we may pick a nonzero element and identify it with $\mathbf{C}$. Then show that β is a nondegenerate symmetric bilinear form on $\bigwedge^2 \mathbf{C}^4$ which is stabilized by $\mathfrak{sl}_4$.

Finally, show that this gives an isomorphism $\mathfrak{sl}_4 \to \mathfrak{so}(\bigwedge^2 \mathbf{C}^4, \beta) \cong \mathfrak{so}_6$.

(10) (Optional) If we pick a symplectic form ω on $\mathbf{C}^4$, we get an evaluation map $f \colon \bigwedge^2 \mathbf{C}^4 \to \mathbf{C}$, namely $f(\sum_i v_i \wedge w_i) = \sum_i \omega(v_i, w_i)$. This is a map of $\mathfrak{sp}_4$-representations if $\mathbf{C}$ is given the trivial action. Using notation from the previous exercise, β restricts to a symmetric bilinear form on $\ker f$; show that it remains nondegenerate and use it to construct an isomorphism $\mathfrak{sp}_4 \to \mathfrak{so}(\ker f, \beta) \cong \mathfrak{so}_5$.