Examples of representations

**Direct products** $G_1, G_2$ groups, $\rho_i : G_i \to GL(V_i)$ reps $i = 1, 2$

$G_1 \times G_2$ has rep on $V_1 \otimes V_2$ by

$$(g_1, g_2) \cdot \sum_{i \in I} v_i \otimes w_i = \sum_{i \in I} (g_1 \cdot v_i) \otimes (g_2 \cdot w_i)$$

$g_1 \in G_1, g_2 \in G_2 \quad v_i \in V_1 \quad w_i \in V_2$

Notation: $V_1 \otimes V_2$ is this rep of $G_1 \times G_2$ external tensor product

$$X_{V_1 \otimes V_2} (g_1, g_2) = X_{V_1} (g_1) \cdot X_{V_2} (g_2)$$

**Facts:**
1. If $V, W$ irreps, then $V \otimes W$ is irrep.
2. If $V_1, \ldots, V_n$ are all irreps. reps of $G_1$, $W_1, \ldots, W_m$ reps of $G_2$, then $\{ V_i \otimes W_j \mid 1 \leq i \leq n, 1 \leq j \leq m \}$ are all irreps reps of $G_1 \times G_2$.

**Abelian groups** Every finite abelian group is isomorphic to direct product of cyclic groups, so suffices to understand cyclic groups.

$G = \mathbb{Z}/m$, let $\omega$ be a primitive $m$th root of unity.

$$[\omega = e^{2\pi i/m}]$$

For $i = 0, \ldots, m-1$ define $\rho_i : \mathbb{Z}/m \to GL(\mathbb{C})$

$$j \to \omega^i$$

These define non-isomorphic reps.

$\Rightarrow$ There are $m$ of them, so we have them all.

**Dihedral groups** For $n \geq 3$, let $D_n$ be symmetry group of a regular $n$-gon ($|D_n| = 2n$)

If $n$ odd, $\Rightarrow \frac{n+3}{2}$ conj. classes
If $n$ even, $\Rightarrow \frac{n+4}{2}$ conj. classes
Center regular n-gon at origin \( \Rightarrow \) \( D_n \) acts by linear transformations, so get rep. on \( \mathbb{R}^2 \) \( \text{[reflection representation]} \)

Can extend scalars to \( \mathbb{C} \), get rep. on \( \mathbb{C}^2 \)

*(exercise: show this is irreducible.)*

**sign representation:** \( g \mapsto \det p(g) \) \( p = \text{reflection rep.} \)

\( D_5: \) 4 conj. classes, size 10,

let \( d_1 \leq d_2 \leq d_3 \leq d_4 \) be dimensions of irreduc. reps.

\( d_1 + d_2 + d_3 + d_4 = 10 \Rightarrow d_1 = d_2 = 1 \rightarrow \text{trivial, sign} \)

\( d_3 = d_4 = 2 \rightarrow \text{reflection V} \)

Need 1 more 2-dim rep.

*guesses:* \( V^* \not\cong V \otimes \text{sign} \)

Recall: \( V^* = \overline{V} \), but \( V \) real-valued, \( \Rightarrow \) \( V \not\cong V^* \)

can show \( V \otimes \text{sign} = X_V \), so \( V \otimes \text{sign} \subseteq V \)

how to build a new 2-dim rep?

Symmetric groups \( S_n = \text{permutations of size } n \)

\( |S_n| = n! \)

**Lemma.** Two permutations are conjugate \( \iff \) have same cycle type.

In particular, \# conj. classes of \( S_n = \# \text{partitions of } n \)

**Proof.** If \( (i_1, i_2, \ldots, i_k) \) denotes cycle \( i_1 \rightarrow i_2 \rightarrow i_3 \rightarrow \cdots \rightarrow i_k \rightarrow i_1 \)

then we have \( T(i_1, i_2, \ldots, i_k) T^{-1} = (T(i_1), T(i_2), \ldots, T(i_k)) \) \( \square \)

\( S_1 = \text{trivial} \checkmark \)

\( S_2 = \mathbb{Z}/2 \)

Assume \( n \geq 3 \)

Every $G_n$ acts on $\mathbb{C}^n$ by permuting coordinates.

If $n \geq 2$, $\mathbb{C}^n$ is reducible: $e_1 + \ldots + e_n$ spans 1-dim subrep.
complement: $\{ (x_1, \ldots, x_n) | x_1 + \ldots + x_n = 0 \}$ standard rep.

exercise: standard rep is irreducible.

Sign rep: $\sigma \rightarrow \text{sgn} (\sigma)$

For $n=2$: $\text{sgn} = \text{standard}$, \{trivial, $\text{sgn}$\} give all irreducible reps.

$G_3$: $|G_3| = 6$, 3 conj. classes.

$d_1 \leq d_2 \leq d_3$ dimensions of irreducible reps.

$d_1^2 + d_2^2 + d_3^2 = 6 \Rightarrow d_1 = d_2 = 1$ trivial, $\text{sgn}$ standard.
$d_3 = 2$ standard.

Character table

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permutation $\sigma$:

$S_3$: $|S_3| = 24$, 5 conj. classes

$d_1, d_2, d_3, d_4, d_5$ be dims of irreducibles reps.

$d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2 = 24$
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Heuristics find set of size 3 w/ action of $G_4$:

$X = \text{set of ways to decompose } (1,3,3,4) \text{ into 2 values} = \{12\mid 34, 13\mid 24, 14\mid 23\}$ $G_4$ acts by

$X \subseteq \{(x,y)\} 
\begin{array}{cccc}
1 & 1 & 0 & 3 \\
2 & 0 & -1 & 2 \\
\end{array}$

$\sigma = (1,2,3,4)$:

$12\mid 34 \rightarrow 23\mid 41 = 14\mid 23$

$13\mid 24 \rightarrow 24\mid 31 = 13\mid 24$

$14\mid 23 \rightarrow 21\mid 34 = 12\mid 34$

$\sigma V \text{ irreducible? yes, can show that } (X_V, X_V) = 1$

or observe $X_V$ not sum of 1-dim characters.