Classification of representations

**Lemma.** Let \( \rho : G \to GL(V) \) be rep, \( f \in CF(G) \). Define
\[
\rho f = \sum_{g \in G} f(g) \rho(g) \quad \text{linear operator on } V.
\]
If \( V \) is irreducible, then \( \rho f \) is scalar = \( \lambda \cdot \text{id}_V \) where
\[
\lambda = \frac{|G|}{\dim V} (f, \overline{\chi_V})_G.
\]

**Proof.** Pick \( h \in G \). Then
\[
\rho(h) \rho f \rho(h)^{-1} = \sum_{g \in G} f(g) \rho(hgh^{-1}) = \sum_{g \in G} f(hgh^{-1}) \rho(g) = \rho f.
\]
\[
\Rightarrow \quad \rho(h) \rho f = \rho f \rho(h) \quad \forall h \in G \Rightarrow \rho f \text{ commutes w/ all elements of } G.
\]
Schur's lemma \( \Rightarrow \) \( \rho f \) is scalar = \( \lambda \cdot \text{id}_V \).
\[
\lambda \dim V = \text{Tr} (\rho f) = \sum_{g \in G} f(g) \chi_V(g) = |G| (f, \overline{\chi_V})_G.
\]
\[
\Rightarrow \quad \lambda = \frac{|G|}{\dim V} (f, \overline{\chi_V})_G. \quad \square
\]

**Thm.**

1. The characters of irreducible reps. form orthonormal basis for \( CF(G) \). In particular, \( \# \) of irreducible reps. of \( G \) (up to isom.) is \( \# \) conj. classes of \( G \).

2. \( V \cong W \iff \chi_V = \chi_W \).

**Proof.**

1. Let \( V, W \) be irreducible reps.
\[
(\chi_V, \chi_W) = \dim \text{Hom}_G(V, W) = \begin{cases} 0 & \text{if } V \not\cong W \\ 1 & \text{if } V \cong W \end{cases}
\]
\[
\Rightarrow \quad \text{If } V_1, V_2, \ldots \text{ are distinct irreducible reps. of } G \text{ (up to isom.)}
\]
then \( \chi_{V_1}, \chi_{V_2}, \ldots \) are orthonormal \( \Rightarrow \) linearly independent.
Need to show: $X_v, \ldots$ span $CF(G)$.

Let $f$ be an element in orthogonal complement of $\text{span}(X_v, \ldots)$

1. $ Pf = 0 \quad \forall \text{irred. reps} \ f : G \to GL(V_i)$

2. $ Pf = 0 \quad \forall \text{reps.} \ f : G \to GL(V)$ (Muschke)

Consider regular rep. $\rho : G \to GL(\mathbb{C}CGJ)$

3. $f = \rho_f(1G) = \sum_{g \in G} f(g) eg \quad f(g) = 0 \quad \forall g \in G$

4. $f = 0$

5. $X_v, \ldots$ spans $CF(G)$

6. $\#\text{irred. reps} = \dim CF(G) = \# \text{conj. classes}$ (up to isom.)

(2) Let $V_1, \ldots, V_c$ be irred. reps of $G$ distinct

For any rep. $V$, $V \cong V_1^{\oplus m_1} \oplus \ldots \oplus V_c^{\oplus m_c}$ (Muschke)

By Schur's lemma, $m_i = (X_{V_i}, X_V)$

If $X_w = X_V \Rightarrow \quad W \cong V_1^{\oplus m_1} \oplus \ldots \oplus V_c^{\oplus m_c} \cong V$.

\[ \square \]

Cor. The multiplicity of an irred. rep $V$ in $\mathbb{C}CGJ$ is $\dim V$

\[ \text{Proof.} \quad \text{multiplicity} = (X_V, X_{\mathbb{C}CGJ}) \]

\[ X_{\mathbb{C}CGJ}(g) = \# \{ h \in G \mid gh = h \} = \begin{cases} 0 & \text{if } g \neq 1_G, \\ |G| & \text{if } g = 1_G \end{cases} \]

\[ (X_V, X_{\mathbb{C}CGJ}) = \frac{1}{|G|} \sum_{g \in G} X_V(1G) = \frac{1}{|G|} \sum_{g \in G} |G| = \dim V \quad \square \]

Cor. Let $d_1, \ldots, d_c$ be dimensions of irred. reps of $G$. Then $d_1^2 + \ldots + d_c^2 = |G|$

\[ \text{Proof.} \quad \mathbb{C}CGJ \cong V_1^{\oplus d_1} \oplus \ldots \oplus V_c^{\oplus d_c} \]

\( |G| = \dim \mathbb{C}[G] = (\dim V_1)^2 + \cdots + (\dim V_c)^2. \)

Cor. If \( G \) is abelian, then every irreducible rep is 1-dimensional.

Pf. \( G \) abelian \( \implies \) all cong. classes are singletons
\[ \implies c = |G| \]
Dimensions of irreducible reps satisfy \( d_1^2 + \cdots + d_c^2 = c \)

Only solution w/ positive integers is \( d_1 = \cdots = d_c = 1. \)