Multilinear Algebra

\( d > 0 \) integer, \( V^{\otimes d} = V \otimes V \otimes \cdots \otimes V \)

\( V^{\otimes 0} = k \)

Given \( \sigma \in S_d \), \( \sigma \cdot (\sum v_1 \otimes \cdots \otimes v_d) = \sum v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)} \) (this gives right action of \( S_d \) on \( V^{\otimes d} \))

\( \text{GL}(V) = \text{invertible linear operators on} \ V \)

For \( g \in \text{GL}(V) \), \( g \cdot (\sum v_1 \otimes \cdots \otimes v_d) = \sum g v_1 \otimes \cdots \otimes g v_d \)

These actions commute.

The \( d \)-th symmetric power of \( V \) is quotient of \( V^{\otimes d} \) by subspace spanned by \( \{ v - \sigma \cdot v \mid v \in V^{\otimes d}, \sigma \in S_d \} \).

This quotient is \( \text{GL}(V) \)-representation.

**Notation:** \( \text{Sym}^d V \or S^d V \)

Given \( v_1 \otimes \cdots \otimes v_d \) where \( v_i \in V \), let \( v_1 \cdots v_d \) denote its image in \( \text{Sym}^d V \). In general, for any \( \sigma \in S_d \), we have

\[ V_1 \cdots V_d = V_{\sigma(1)} \cdots V_{\sigma(d)} \]

**Intuition:** \( \text{Sym}^d V = \text{space of degree} \, d \, \text{polynomials} \)

For \( g \in \text{GL}(V) \), \( g \cdot (v_1 \cdots v_d) = (g v_1) \cdots (g v_d) \).

let \( e_1, \ldots, e_n \) be basis for \( V \).

Then \( \{ e_{i_1} \otimes \cdots \otimes e_{i_d} \mid i_1 < \cdots < i_d \in \{1, \ldots, n\} \} \) is basis for \( V^{\otimes d} \).

Claim: \( \{ e_{i_1} \otimes \cdots \otimes e_{i_d} \mid 1 \leq i_1 \leq \cdots \leq i_d \leq n \} \) is basis for \( \text{Sym}^d V \)

**Pf.** Spanning is easy.

Picks scalars \( \alpha_1, \ldots, \alpha_n \). Let \( g = (\alpha_1 \cdots 0 \cdots \alpha_n) \in \text{GL}_n(k) \)

\( \text{GL}_n(k) \)
\[ g_{e_1 \ldots e_d} = \alpha_1 \ldots \alpha_d e_1 \ldots e_d. \]

So \( e_1, \ldots, e_d \) are eigenvectors. If \( K \) is infinite, then we can pick \( \alpha_1, \ldots, \alpha_d \) so that \( \alpha_1, \ldots, \alpha_d \) are all distinct.

If not, we can extend coefficients (this does not affect linear independence).

Consider linear functional on \( V^d \) that sends each \( e_i, \ldots, e_d \) to 1. This factors through to linear functional on \( \text{Sym}^d V \) that values 1 on each \( e_1, \ldots, e_d \) (so they're \( \neq 0 \)).

\[ \text{dim} \text{ Sym}^d V = \binom{n+d-1}{d}, \quad n = \text{dim} V. \]

Given \( d, e \geq 0 \), we have "multiplication" map

\[ \mu: \text{Sym}^d V \otimes \text{Sym}^e V \to \text{Sym}^{d+e} V \]

\[ (v_1, \ldots, v_d) \otimes (w_1, \ldots, w_e) \mapsto v_1 \ldots v_d w_1 \ldots w_e \]

\( GL(V) \)-equivariant.

The \( d \)-th exterior power of \( V \) is quotient of \( V^d \) by the \( d \)-th exterior power of \( V \) is quotient of \( V^d \) by

subspace spanned by \( \{ v_1 \otimes \cdots \otimes v_d \mid \exists i \neq j \text{ s.t. } v_i = v_j \}. \)

Notation: \( \wedge V \). This is \( GL(V) \)-representation.

Given \( v_1 \otimes \cdots \otimes v_d \), \( v_i \in V \), let \( v_1 \wedge \cdots \wedge v_d \) be its image in \( \wedge V \). For \( g \in GL(V) \),

\[ g \cdot (v_1 \wedge \cdots \wedge v_d) = (gv_1) \wedge \cdots \wedge (gv_d). \]
Rank. Note: \( \text{V}_1 \wedge \ldots \wedge \text{V}_d \) is skew-symmetric in the sense that
\[
\text{V}_i \wedge \ldots \wedge \text{V}_d = \text{sgn}(\sigma) \text{V}_{\sigma(1)} \wedge \ldots \wedge \text{V}_{\sigma(d)}
\]
for any \( \sigma \in \text{S}_d \).

When \( d = 2 \):
\[
0 = (\text{V}_1 + \text{V}_2) \wedge (\text{V}_1 + \text{V}_2) = \text{V}_1 \wedge \text{V}_1 + \text{V}_1 \wedge \text{V}_2 + \text{V}_2 \wedge \text{V}_1 + \text{V}_2 \wedge \text{V}_2
\]

\[
\Rightarrow \quad \text{V}_1 \wedge \text{V}_2 = -\text{V}_2 \wedge \text{V}_1
\]

We could define \( \Lambda V \) as quotient by \( \{ v \sim \text{sgn}(\sigma) \sigma v \} \)

if \( \text{char}(K) \neq 2 \):

for any \( v \in V \), \( \Lambda^k v = -v \wedge v \Rightarrow 2v \wedge v = 0 \)

Pick basis \( e_1, \ldots, e_n \) for \( V \). Then
\[
\{ e_{i_1} \wedge \ldots \wedge e_{i_d} \mid 1 \leq i_1 < \ldots < i_d \leq n \}
\]
is basis for \( \Lambda^d V \).

\[
\Rightarrow \quad \dim \Lambda^d V = \binom{n}{d}
\]

\( \text{Note: this is 0 if } d > \dim V. \)

Multiplication map:
\[
\mu: \Lambda^d V \otimes \Lambda^d V \rightarrow \Lambda^d V
\]
\[
(v_1 \wedge \ldots \wedge v_d) \otimes (w_1 \wedge \ldots \wedge w_n) \rightarrow v_1 \wedge \ldots \wedge v_d \wedge w_1 \wedge \ldots \wedge w_n
\]

\( \mu \) is \( GL(V) \)-equivariant.

Comultiplication map:
\[
\Delta: \Lambda^d V \rightarrow \Lambda^d V \otimes \Lambda^d V
\]
\[
v_1 \wedge \ldots \wedge v_d \rightarrow \sum \text{sgn}(I, I^c) v_I \otimes v_{I^c}
\]
\[I \subseteq \text{subsets of } 1, \ldots, d, \text{ of size } d\]
\[
V_I = v_{i_1} \wedge \ldots \wedge v_{i_d} \text{ if } i_1 < \ldots < i_d \text{ are elements of } I
\]
\[
V_{I^c} = \ldots
\]
$\text{sgn} (I, I^c) \in \mathbb{Z}/2, -1$ satisfies $v_{\text{L}} \wedge v_{\text{R}} = \text{sgn}(\sigma_{\text{L}}) v_{\sigma_{\text{L}}} \wedge \ldots \wedge v_{\sigma_{\text{R}}}.$

$\Delta$ is $\text{GL(U)}$-equivariant.

$\Delta$ is "coassociative" compositions are same:

$$
\Delta v \circ f = \text{det } f \cdot \Delta v \otimes \Delta v
$$

$$
\wedge v \rightarrow \wedge v \otimes \wedge v \rightarrow \wedge v \otimes \wedge v \otimes \wedge v
$$

If we iterate $\Delta$ $d-1$ times, get to

$$
\wedge v \rightarrow v \otimes d \wedge v \quad \text{(since } \wedge v = v)\]

$$

$v \wedge \ldots v \wedge v \rightarrow \sum_{\sigma \in S_d} (\text{sgn } \sigma) v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(d)}$.

This is injective, so $\wedge v$ is a subrepresentation of $v \otimes d$.

(The composition $\wedge v \rightarrow v \otimes d \rightarrow \wedge v$

is $d!$ times identity on $\wedge v$.

So if characteristic larger than $d$, then $\wedge v$ is a direct

summand of $v \otimes d$.

The image is subspace $\{v \in v \otimes d \mid v = \text{sgn } \sigma (v) \wedge v \sigma \in S_d \}$.

The $d$th divided power is subspace $\{v \in v \otimes d \mid v = \sigma v \wedge v \sigma \in S_d \}$.

The $d$th divided power is $\text{GL(U)}$-subrep.

- $D^d v$ is generally not isomorphic to $\text{Sym}^d v$
  (but have same dim). Holds if char $k = 0$ or larger than $d$.
- In general, $D^d (v^*) \cong (\text{Sym}^d v)^*$ as $\text{GL(U)}$-representations.
For $d=1$, $V^\otimes 1 = V$ is irreducible representation of $GL(V)$.

For $d=2$, if char $k \neq 2$, we can decompose $V^\otimes 2$:

\[
V^\otimes 2 \cong \bigwedge^2 V \oplus \text{Sym}^2 V
\]

and

\[
(v \otimes w) \mapsto \left( \frac{v \otimes w - w \otimes v}{2}, \frac{v \otimes w + w \otimes v}{2} \right)
\]

For $d=3$, there's more to $V^\otimes 3$ than just $\bigwedge^3 V$ & $\text{Sym}^3 V$:

[If $\dim V = 2$, then $\dim V^\otimes 3 = 8$
but $3V = 0$ and $\dim \text{Sym}^3 V = 4$]