Littlewood–Richardson coefficients

\[ C^\nu_{\lambda \mu} \] \( \lambda, \mu, \nu \) partitions \( (\nu) = m \nu \)
\[ | \lambda | = m \]
\[ | \mu | = n \]

They appear in these contexts:

- Multiplication of Schur functions:
  \[ S^\lambda \cdot S^\mu = \sum C^\nu_{\lambda \mu} S^\nu \]

- Expansion of skew Schur function:
  \[ S_{\nu/\mu} = \sum C^\lambda_{\nu/\mu} S^\lambda \]

- Induction of Specht modules:
  \[ \text{Ind}_{G_{|\lambda|+|\mu|}}^{G_{|\lambda|+|\mu|+1}} (S^\lambda \otimes S^\mu) \cong \bigoplus \binom{\nu}{\lambda} C^\nu_{\lambda \mu} S^\nu \]

- Restriction of Specht modules
  \[ \text{Res}_{G_{|\lambda|}}^{G_{|\lambda|+|\mu|}} (S^\lambda \otimes S^\mu) \cong \bigoplus C^\nu_{\lambda \mu} S^\nu \]

Let \( w = w_1 w_2 \ldots w_n \) sequence of positive integers.

Let \( m_i(w) = \# j \mid w_j = i \)

A prefix of \( w \) is sequence \( w_1 \ldots w_m \) for some \( m \leq n \).

\[ m_i(w) \]

**Def.** \( w \) is a lattice permutation / Yamanouchi word / ballot sequence

- if, for every prefix \( v \) of \( w \), we have \( m_i(v) \geq m_{i+1}(v) \) for all \( i \).

**Def.** \( T \) = tableau, its reverse reading word is sequence of its entries reading right to left starting from first row, and moving down.

**Def.** A SSYT \( T \) is a L-R tableau if its reverse reading word is a lattice permutation.

**Thm.** \( C^\nu_{\lambda \mu} = \# 

**Ex.** \( \lambda = (4, 2, 1) \), \( \mu = (5, 2) \), \( \nu = (6, 5, 2, 1) \)

\[ C^\nu_{\lambda \mu} = 3 \]
Remark: This rule generalizes Pieri's rule:

Suppose $\lambda = (d)$. Then $C^{(d)}_{\mu} = \# L-R tableaux of shape $\mu^d$ of type $(d)$, i.e., only using $1's$.

rem (in the rule): no two boxes in same column

\[ C^{(d)}_{\mu} = \begin{cases} 1 & \text{if $\mu^d$ is horizontal strip of size $d$} \\ 0 & \text{otherwise} \end{cases} \]

Now suppose $\lambda = (1^d)$. Then $C^{(1^d)}_{\mu} = \# L-R tableaux of shape $\mu^d$ of type $(1^d)$, i.e., using $1, \ldots, d$ each exactly once.

reverse reading word must be $12\ldots d$

\[ C^{(1^d)}_{\mu} = \begin{cases} 1 & \text{if $\mu^d$ is a vertical strip} \\ 0 & \text{otherwise} \end{cases} \]

This rule shows:

\[ C^{\nu}_{\lambda^d} > 0 \iff \text{for any integer } d > 0, \]

\[ d \nu > d \lambda^d > 0. \]

$C^{\nu}_{\lambda^d} > 0$ \iff there exists $d > 0$ s.t. $\frac{d\nu}{d\lambda^d} > 0$.

yes, "saturation property"