Schensted insertion

\( T = SSYT \) of shape \( \lambda \), \( k \) positive integer

Row/Schensted insertion \( T \prec k \) is defined by:

1. Find largest index \( i \) s.t. \( T_{i,i-1} \leq k \). (If it doesn't exist set \( i = 1 \)).
2. Replace \( T_{i,i} \) w/ \( k \). If \( i = \lambda_1 + 1 \), then this means add box to end of first row of \( \lambda \) and put in value \( k \). In this case, result is \( T \prec k \), finish. Else, let \( k' = \) old value of \( T_{i,i} \) (\( k \) bumps \( k' \)). Move to next step.
3. Let \( T' = \) remove first row of \( T \) from \( T \).
   
   Compute \( T' \prec k' \). Add back new first row of \( T \) to \( T \prec k' \).

Insertion path \( I(T \prec k) \) is coordinates of bumped boxes.

Ex.: \( T = \begin{array}{cccccccc}
1 & 2 & 4 & 5 & 5 & 6 \\
3 & 3 & 6 & 6 & 8 & k = \mathbf{4} \\
4 & 6 & 8 \\
7 \\
9 \\
\end{array} \)

    \( I(T \prec 4) \) is \( \begin{array}{cccccccc}
1 & 2 & 4 & \mathbf{5} & 5 & 6 \\
3 & 3 & 6 & 6 & 8 \\
4 & 6 & 8 \\
7 \\
9 \\
\end{array} \)


\[ T_{\leq k} = \begin{array}{cccccc}
1 & 2 & 4 & 4 & 5 & 6 \\
3 & 3 & 5 & 6 & 8 \\
4 & 6 & 8 \\
7 & 8 \\
9 \\
\end{array} \quad I(T_{\leq k}) = \{(1,4), (2,3), (3,3), (4,2)\}
\]

**Prop** \( T_{\leq k} \) is SSYT.

**Pf.** By construction, rows are weakly increasing.

**Claim:** If \((i,j), (i+1,j') \in I(T_{\leq k})\), then \(j \leq j'\).

If not, then have \( (i,j) \) \( \square \)

But \( T_{i,j} < T_{i+1,j} \) and \( T_{i,j} \) is being inserted into row \( i+1 \).

This violates construction \( \square \).

In particular, let \( T' = T_{\leq k} \). For each \((i,j) \in I(T_{\leq k})\), we need to check \( T'_i,j \leq T'_i,j' \leq T'_{i+1,j} \).

**Note:** If \( T'_{i,j} \) is value bumped from row \( i \) or \( i-1 \), this was bumped by something smaller, i.e., \( T'_{i,j} > T'_{i-1,j} \) where \((i-1,j') \in I(T_{\leq k})\). By claim, \( j' \geq j \), so \( T'_{i-1,j} \geq T'_{i,j} \).

1. Let \((i+1,j') \in I(T_{\leq k})\)
   \( T'_{i,j} < T'_{i,j'} \). If \( j = j' \), then \( T_{i,j} = T'_{i+1,j} \) \( \checkmark \)
   Else, \( j' < j \), so \( T'_{i+1,j} = T'_{i,j'} \) \( \checkmark \)

**Lemma** ("Nested property") Let \( T = SSYT, j \leq k \).

Then \( I(T_{\leq j}) \) is strictly to left of \( I(T_{\leq j}, k) \)

i.e., if \( (r,s) \in I(T_{\leq j}) \) then \( r \leq r' \),

Furthermore, \# \( I(T_{\leq j}) \) \( \geq \) \# \( I(T_{\leq j}, k) \).
pf. When inserting \( k \) into first row of \( T \leq j \), position that \( k \) bumps is strictly to the right of position that \( j \) bumps. If \( j \) bumps \( j' \), \( k \) bumps \( k' \), then \( j' \leq k \), so rest of first statement follows by recursive nature of row insertion.

For second statement, let \( r = \# I(T \leq j) \). Suppose \( r \leq \# I(T \leq j) \leq k \).

Then, consider insertions in row \( r \): for \( T \leq j \), value gets added to the end. For \( (T \leq j) \leq k \), insertion happens to right of that, i.e., the end of the row, so \( r = \# I(T \leq j) \leq k \).

\[ \square \]

**Lemma.** \( T \leq SSY \), \( j \geq k \). If \( (r,s) \) is last box of \( I(T \leq j) \) and \( (r',s') \) last box in \( I(T \leq j) \leq k \), then \( s' \leq s \).

**pf.** Suppose \( j \) is added to end of first row of \( T \).

Since \( k \leq j \), it must bump something in first row of \( (T \leq j) \) and so final box must appear weakly to left of end of first row of \( (T \leq j) \).

Otherwise, \( j \) bumps \( j' \) & \( k \) bumps \( k' \) & \( k' \leq j' \leq j \) and so can reduce to smaller \( SSY \) by removing first row.

**"Reversing" row insertion:**

Given \( SSY \) \( T \) and location a value \( k \) which is at end of its row, we can undo row insertion:

\exists! \ SSY \ T \ & \text{value } i \text{ s.t. } T' = (T \leq i) \text{ and } k \text{ is last value inserted.}