Frobenius characteristic map

\[ CF_n = \mathbb{Q}-\text{valued class functions on } S_n \]

\[ (\varphi, \psi)_{S_n} = \frac{1}{n!} \sum_{\sigma \in S_n} \varphi(\sigma) \psi(\sigma) \]

Given \( \sigma \in S_n \), let \( t(\sigma) \) = partition whose parts are cycle lengths of \( \sigma \) (cycle type of \( \sigma \)).

\[ \lambda = \text{partition, } m_i(\lambda) = \# \text{ of times } i \text{ appears in } \lambda \]

\[ \varepsilon_\lambda = \prod_{i} m_i(\lambda)! \cdot i \]

**Lemma.** Conjugacy class of \( \lambda \) has size \( \frac{h_!}{\varepsilon_\lambda} \).

**Proof.** Given \( \sigma \), \( t(\sigma) = \lambda \), suffices to show centralizer of \( \sigma \) has size \( \varepsilon_\lambda \).

Note: each cycle \((i_1, i_2, \ldots, i_k)\) is equal to cyclic shift \((i_k, i_1, \ldots, i_{k-1})\).

There are \( \prod_{i} m_i(\lambda)! \) many ways to permute each cycle cyclically.

Thus, \( \Rightarrow \) \( \varepsilon_\lambda \) many elements in centralizer.

Let \( 1_\lambda \) be indicator function on conjugacy class of \( \lambda \)

i.e., \( 1_\lambda(\sigma) = \begin{cases} 1 & \text{if } t(\sigma) = \lambda \\ 0 & \text{else} \end{cases} \)

**Cor.** \( (\lambda, 1_\mu)_{S_n} = \frac{1}{h_!} \delta_{\lambda\mu} \)

**Proof.** If \( \lambda \neq \mu \), then \( (\lambda, 1_\mu)_{S_n} = 0 \)

Otherwise, \( (\lambda, 1_\lambda)_{S_n} = \frac{1}{h_!} \cdot c \) where \( c = \text{size of conjugacy class.} \)
Given $n,m$. Think of $G_n \times G_m$ as subgroup $G_{n+m}$:

$G_n = \{ \sigma \in G_{n+m} | \sigma(i) = i \text{ for } i = n+1, \ldots, n+m \}$

$G_m = \{ \sigma \in G_{n+m} | \sigma(i) = i \text{ for } i = 1, \ldots, n \}$

Define induction product:

$\psi : CF_n \times CF_m \rightarrow CF_{n+m}

\psi \circ \psi = \text{Ind}_{G_{n+m}}^{G_{n \times G_m}} (\psi \otimes \psi)

This gives $CF = \bigoplus_{n \geq 0} CF_n$ a ring structure.

Frobenius characteristic map:

$\text{ch} : CF_n \rightarrow \Lambda \Phi_n$

$\text{ch} (\psi) = \frac{1}{n!} \sum_{\sigma \in S_n} \psi(\sigma) \rho_\lambda(\sigma)$

$= \sum_{\lambda \in \text{Part}(n)} z_\lambda^{-1} \psi(\lambda) \rho_\lambda$

Combine then to get $\text{ch} : CF \rightarrow \Lambda \Phi$

Prop. $\text{ch}$ is an isometry: $(\psi, \psi)_{G_n} = \langle \text{ch}(\psi), \text{ch}(\psi) \rangle$

for all $\psi, \psi \in CF_n$.

Pf. $\langle \text{ch}(\psi), \text{ch}(\psi) \rangle = \langle \sum_{\lambda} z_\lambda^{-1} \psi(\lambda) \rho_\lambda, \sum_{\mu} z_\mu^{-1} \psi(\mu) \rho_\mu \rangle$

$= \sum_{\lambda} z_\lambda^{-1} \psi(\lambda) \psi(\lambda) = (\psi, \psi)_{G_n}$
Prop. $\chi$ is a ring homomorphism: $\chi(\psi \circ \psi') = \chi(\psi) \chi(\psi')$.

Pf. Given $\lambda, \mu$, let $\lambda \vee \mu$ be partition obtained by taking union of all parts in order.

Claim: $\chi \circ \lambda \mu = \frac{\chi \lambda \mu}{2 \times 2 \mu}$

Let $\nu$ be partition of $\lambda \mu \nu$. Then:

\[
\left( \text{Ind} \left[ G_{\chi \lambda \mu \nu}^{G_{\chi \lambda \mu \nu}} \right] \right)_{\chi \lambda \mu \nu}(1_{\chi \lambda \mu \nu}) = \left( \chi \lambda \mu \nu, \text{Res} \left[ G_{\chi \lambda \mu \nu}^{G_{\chi \lambda \mu \nu}} \right] \right)_{\chi \lambda \mu \nu}(1_{\chi \lambda \mu \nu}) = \frac{\delta_{\chi \lambda \mu \nu}}{2 \times 2 \mu}
\]

Write $\chi \circ \lambda \mu = \sum_{\nu} c_{\chi \lambda \mu} \nu$. Then

\[
(\chi \circ \lambda \mu)_{\chi \lambda \mu \nu} = c_{\chi \lambda \mu} \nu /
\]

\[
\Rightarrow c_{\chi \lambda \mu} = \frac{z_{\chi \lambda \mu} \delta_{\chi \lambda \mu \nu}}{2 \times 2 \mu}
\]

\[
\Rightarrow \chi \circ \lambda \mu = \frac{z_{\chi \lambda \mu} \chi \lambda \mu}{2 \times 2 \mu}.
\]

\[
\chi(\chi) = \frac{p_{\chi}}{2 \times 2 \chi}, \quad \text{so}
\]

\[
\chi(\chi) = \frac{z_{\chi \lambda \mu} \chi(\chi) \lambda \mu}{2 \times 2 \mu}
\]

\[
\chi(\chi) \chi(\chi) = \frac{p_{\chi} \lambda \mu}{2 \times 2 \mu}
\]

\[
\text{So} \quad \chi(\chi \circ \lambda \mu) = \chi(\chi) \chi(\chi) \chi(\chi)
\]

In general, if $\psi = \sum_{\lambda} \psi(\lambda) \chi(\lambda), \psi = \sum_{\mu} \psi(\mu) \chi(\mu)$,

\[
\text{then} \quad \chi(\psi \circ \psi') = \sum_{\lambda} \chi(\psi(\lambda) \psi(\lambda')) \chi(\lambda) \chi(\lambda') \chi(\psi(\lambda)) \chi(\psi(\lambda'))
\]

\[
\chi(\psi(\lambda) \psi(\lambda')) = \sum_{\lambda'} \chi(\psi(\lambda)) \chi(\psi(\lambda'))
\]

\[
\chi(\psi(\lambda)) \chi(\psi(\lambda')) = \chi(\psi(\lambda)) \chi(\psi(\lambda')).
\]
Finally, $\Lambda_{\mathfrak{a}}$ map to a basis for $\Lambda_{\mathfrak{a}}$, so $\mathfrak{a}$ is an isomorphism. 

Let $CF'_n \subset CF_n$ be abelian subgroup spanned by character of representations. ($CF'_n$ is free abelian group w/ basis characters of irreducible representations = Specht modules)

Note: $CF' = \bigoplus_{n \geq 0} CF'_n$ is subring of $CF$ under $\circ$.

Let $1_{G_n}$ be trivial character: $1_{G_n}(\sigma) = 1$ for all $\sigma$.

For partition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, define $o_{\alpha_1} \circ \cdots \circ o_{\alpha_k} : 1_{G_d} \to 1_{G_{\alpha_1} \times \cdots \times G_{\alpha_k}}$.

Recall: $M^\alpha = \text{permutation representation on } d$-tadroids

$S^\alpha = \text{Specht module}$

Prop. $\eta^\alpha = \text{character of } M^\alpha$. So $\eta^\alpha \in CF'$

Furthermore, $\eta^\alpha$ form basis for $CF'$.

If. From before: $M^\alpha = \text{Ind}_{G_d \times \cdots \times G_d}^{G_n} (\text{trivial})$

Also: if $\exists$ nonzero $G_n$-equivariant map $S^\lambda \to M^\mu$, then $\lambda \geq \mu$

and $S^\lambda$ appears once in decomposition of $M^\lambda$ into irreducible reps.

$\Rightarrow$ if $\eta^\lambda = \sum_{\beta} a_{\beta} \eta^\beta$ (character of $S^\beta$)

then $a_{\beta} \to 0 \Rightarrow \beta \geq \lambda \Rightarrow (a_{\beta})$ invertible over $\mathbb{N}$

and $a_{\alpha \lambda} = 1$ $\Rightarrow$ $\eta^\lambda$ $|\alpha| = d$ basis for $CF'_d$. 

Prop. \( \text{ch}(\eta^d) = h_\alpha \). In particular, \( \text{ch}(C_{\ell}) = \Lambda \).

Pf. \( \text{ch}(\eta^d) = \sum \chi_\alpha^{-1} r_\alpha = h_\alpha \)

Since \( \text{ch} \) is ring homomorphism, \( \text{ch}(\eta^d) = h_\alpha. \)

Prop. Suppose \( \psi_1, \ldots, \psi_{\ell} \in C_{\ell} \) are an orthonormal basis. Then irreducible characters must be \( \pm \psi_1, \ldots, \pm \psi_{\ell} \).

Pf. Write \( \psi_i \) as integer linear combinations of irreducible characters. These coefficients give orthogonal matrix \( A \), i.e., \( AA^T = \text{id} \).

Each row is unit vector \( \Rightarrow \pm \) standard basis vector.

\( \Rightarrow A \) is permutation matrix up to signs.

Goal: Find orthonormal basis for \( \Lambda \) wrt. \( \langle \cdot, \cdot \rangle \).