The involution $\omega$

Since $e_1, e_2, \ldots$ are independent generators for $\Lambda$, we can define any homomorphism $f: \Lambda \rightarrow R$ (R-commuting) by picking $f(e_1), f(e_2), \ldots$ arbitrarily.

Every $r \in \Lambda$ is uniquely of the form $\sum_{k} c_\chi e_{\chi,k}$

so $f(r) = \sum_{k} c_\chi f(e_{\chi,k}) = f(e_{\chi,k})$

Define $\omega: \Lambda \rightarrow \Lambda$ by $\omega(e_i) = h_i$ for $i \geq 1$

where $h_n = \sum_{i_1 \leq \cdots \leq i_n} x_{i_1} x_{i_2} \cdots x_{i_n}$ (sum of all monomials of degree $n$)

Then $\omega^2 = \text{id}$, equivalently, $\omega(h_n) = e_n$ for all $n \geq 1$.

pf. Consider $\Lambda[[t]] = \text{power series w/ coefficients in } \Lambda$.

Define $E(t) = \sum_{n \geq 0} e_n t^n$, $H(t) = \sum_{n \geq 0} h_n t^n$

We have $E(t) = \prod_{i \geq 1} (1 + x_i t)$

$H(t) = \prod_{i \geq 1} (1 + x_i t + x_i^2 t^2 + \cdots) = \prod_{i \geq 1} (1 - x_i t)^{-1}$

$\Rightarrow E(t) H(-t) = 1$

$\uparrow$ $n$th coefficient is 0 for $n > 0$

$0 = (1) \sum_{i=0}^{\infty} e_i(1)^i h_{n-i}$

Apply $\omega$ $0 = \sum_{i=0}^{\infty} e_i h_i \omega(h_{n-i})$

$\Rightarrow \sum \omega(h_n) t^n$ is inverse of $H(-t)$

$\Rightarrow \sum_{n \geq 0} \omega(h_n) t^n = E(t) \Rightarrow \omega(h_n) = e_n$ for $n > 1$. $\square$
\( \Lambda(n) \) is polynomial ring w/ generator \( e_1(x_1, \ldots, x_n), \ldots, e_n(x_1, \ldots, x_n) \)

Define \( \omega_n : \Lambda(n) \rightarrow \Lambda(n) \) by \( \omega_n(e_i) = h_i \).

\[ \omega_n^2 = \text{id}, \text{ equivalently, } \omega_n(h_i) = e_i \text{ for } i = 1, \ldots, n. \]

**Thm.** \( \omega_n = \text{id} \).

**Pf.** Define \( E_n(t) = \sum_{i=0}^{n} e_i(x_1, \ldots, x_n) t^i \), \( H_n(t) = \sum_{i=0}^{n} h_i(x_1, \ldots, x_n) t^i \).

\[ \Rightarrow E_n(t) = \prod_{i=1}^{n} (1 + x_i t), \quad H_n(t) = \prod_{i=1}^{n} (1 + x_i t)^{-1}. \]

\[ \Rightarrow E_n(t) H_n(-t) = 1 \]

\[ \text{min}(k,n) \]

\[ \Rightarrow \sum_{i=0}^{\min(k,n)} e_i(x_1, \ldots, x_n) t^i \prod_{j=1}^{k-i} h_j^{-1}(x_1, \ldots, x_n) = 0 \text{ for } k > 0. \]

\[ \omega_n \]

\[ \sum_{i=0}^{\min(k,n)} h_i(x_1, \ldots, x_n) t^i \prod_{j=1}^{k-i} \omega_n(h_j^{-1}(x_1, \ldots, x_n)) = 0 \]

In particular, \( \sum_{i=0}^{\min(k,n)} \omega_n(h_i(\ldots, x_n)) t^i \) agrees w/ inverse of \( H(-t) \) up to degree \( n \). \( \Rightarrow \omega_n(h_i) = e_i \quad \text{for } i = 1, \ldots, n. \)

Complete homogeneous symmetric functions

For partition \( \lambda = (\lambda_1, \ldots, \lambda_k) \) define

\[ h_{\lambda} = h_{\lambda_1} \cdots h_{\lambda_k} \]

\[ \Rightarrow \omega(h_{\lambda}) = h_{\lambda}. \]

\[ \text{Thm. } \{h_{\lambda} \mid \lambda \in \text{Par} \} \text{ is basis for } \Lambda. \]

**Pf.** \( \omega \) is isom., and \( \{e_\lambda \mid \lambda \in \text{Par} \} \) is a basis. \( \square \)
Note: \( \exists \text{ coeff } N_{\lambda, \mu} \text{ s.t.} \)

\[ h_{\lambda} = \sum_{\mu} N_{\lambda, \mu} m_{\lambda, \mu}. \]

\( N_{\lambda, \mu} \) = \# of matrices \( A \) w/ non-negative integer entries
s.t. \( \text{row}(A) = \lambda, \text{col}(A) = \mu. \)

\( h, (x_1, \ldots, x_n), \ldots, h, (x_1, \ldots, x_n) \) are alg. independent generators
for \( \Lambda(n) \) and \( \{ h_{\lambda}(x_1, \ldots, x_n) \mid \lambda \leq n, |\lambda| = d \} \)
is a basis for \( \Lambda(n)_{\mathbb{A}}. \)