Symmetric Functions

$x_1, \ldots, x_n$ finite set of variables

$\mathbb{Z}[x_1, \ldots, x_n] =$ polynomials w/ $\mathbb{Z}$-coefficients

$S_n$ acts by permuting variables.

**Ring of symmetric polynomials:** $\Lambda(n) := \mathbb{Z}[x_1, \ldots, x_n] / S_n$

$\Lambda(n) = \{ f \in \mathbb{Z}[x_1, \ldots, x_n] \mid \sigma.f = f \ \forall \sigma \in S_n \}$

$\Lambda(n)$ subring of $\mathbb{Z}[x_1, \ldots, x_n]$

Consider $x_1, x_2, \ldots$ countable set of variables.

$S_\infty =$ permutations of $\{1, 2, \ldots \}$

$R =$ set of power series in $x_1, x_2, \ldots$ of bounded degree w/ $\mathbb{Z}$-coeff.

$S_\infty$ acts on $R$ by permuting variables

$\Lambda := R / S_\infty = \{ f \in R \mid \sigma.f = f \ \forall \sigma \in S_\infty \}$

ring of symmetric functions.

$R$ is a ring, $\Lambda$ is a subring.

For every $n$, have $\pi_n : \Lambda \rightarrow \Lambda(n)$

$f \rightarrow f(x_1, \ldots, x_n, 0, 0, \ldots )$

For each integer $d \geq 0$, let

$\Lambda(n)_d = \{ f \in \Lambda(n) \mid f \text{ homogeneous of degree } d \}$

$\Lambda_d = \{ f \in \Lambda \mid f \text{ homogeneous of degree } d \}$

$\Lambda(n) = \bigoplus_{d \geq 0} \Lambda(n)_d$,  \hspace{1cm} \Lambda = \bigoplus_{d \geq 0} \Lambda_d$

let $\Lambda(n)$ be symmetric polynomials in $x_1, \ldots, x_n$ w/ $\mathbb{Z}$-coeff.

\Lambda be Symmetric functions w/ $\mathbb{Q}$-coeff.
\[ \text{Ex. } \quad p_k := \sum_{i \geq 1} x_i^k = x_1^k + x_2^k + \ldots \quad \text{(power sum)} \]

\[ e_k := \sum_{1 \leq i_1 < i_2 < \ldots < i_k} x_{i_1} x_{i_2} \ldots x_{i_k} \quad \text{(elementary)} \]

\[ h_k := \sum_{i_1 \leq i_2 \leq \ldots \leq i_k} x_{i_1} x_{i_2} \ldots x_{i_k} \quad \text{(complete homogeneous)} \]

\[ e_1 = h_1, \quad e_2 = x_1 x_2 + x_1 x_3 + \ldots, \quad h_2 = e_2 + p_2 \]

\[ h_3 = e_3 + p_3 + \sum_{i \neq j} x_i^2 x_j \]

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**Monomial symmetric functions**

Given sequence \((x_1, x_2, \ldots)\), finitely many nonzero entries

\[ x^\alpha := \prod_{i \geq 1} x_i^{a_i} = x_1^{a_1} x_2^{a_2} \ldots \]

Given partition \(\lambda\), define

\[ m_\lambda := \sum_{\alpha} x^\alpha \quad \text{permutations of } (\lambda_1, \lambda_2, \ldots) \quad \text{(sum only over distinct values)} \]

**Ex.**

\[ m_1 = x_1 + x_2 + \ldots = p_1 = e_1 = h_1 \]

\[ m_{1,1} = \sum_{i < j} x_i x_j = e_2 \]

In general, \(m_\lambda = e_\lambda\) \& \(m_k = p_k\)

\[ m_{321} = \sum_{i,j,k} x_i^3 x_j^2 x_k \]

\[ i \neq j, \quad i \neq k, \quad j \neq k \]
Thm. \( \{ \lambda \} \) is a basis for \( \Lambda \).

\[
\text{Pf. Linear independence: no } 2 \text{ } \lambda \text{'s share common monomial Span: given } f \in \Lambda, \text{ write } f = \sum c_\lambda x^\lambda, \quad c_\lambda \in \mathbb{Z} \\
& \text{ s.t. } |\lambda| \leq d \\
& \text{if } \alpha \text{ is permutation of } \beta, \text{ so can write } f = \sum c_\lambda x^\lambda \\
& \text{finite sum partitions} \text{ partitions} \rightarrow \text{ finite sum} \quad \square
\]

Cor. \( \Lambda \) has basis \( \{ \lambda \} \mid |\lambda| = d \}, \text{ hence is free abelian group of rank } \text{pld}).

\( \Lambda(n)_d \) has a basis \( \{ \lambda(x_1, \ldots, x_n) \mid |\lambda| = d, \ell(\lambda) \leq n \} \).

Elementary symmetric functions

For partition \( \lambda = (\lambda_1, \ldots, \lambda_k) \), define

\[
e_\lambda = e_{\lambda_1} \cdots e_{\lambda_k}
\]

\( e_\lambda \in \Lambda(1x_1) \).

\( \exists \text{ coeff. } M_{\lambda \mu} \in \mathbb{Z} \text{ s.t.} \)

\[
e_\lambda = \sum M_{\lambda \mu} \mu_m
\]

Given infinite matrix \( A \) with finitely many nonzero entries,

\[
\text{row}(A) = \left( \sum_{i \geq 1} A_{i1}, \sum_{i \geq 1} A_{i2}, \ldots \right)
\]

\[
\text{col}(A) = \left( \sum_{j \geq 1} A_{j1}, \sum_{j \geq 1} A_{j2}, \ldots \right)
\]

\( A \) is \((0,1)\)-matrix if every entry is 0 or 1.
Lemma. \( M_{\lambda, \mu} \) is \# of \((0,1)\)-matrices w/ \( \text{row}(A) = \lambda \), \( \text{col}(A) = \mu \).

Proof. \( e_{\lambda} = e_{\lambda_1}e_{\lambda_2} \cdots e_{\lambda_k} = (\sum x_{i_1} x_{i_2} \cdots x_{i_{\lambda_1}})(\sum x_{j_1} \cdots x_{j_{\lambda_2}}) \cdots \)

\[ M_{\lambda, \mu} = \text{coeff. of } x^\mu \text{ in } e_{\lambda} \]

= sum over all choices of monomials \( x_i \) whose product is \( x^\mu \)

Given monomial \( x_i \), encode as sequence \((a_1, a_2, \ldots)\)

where \( a_i = \text{exponent of } x_i \)

gives \((0,1)\)-matrix \( A \) by letting row \( i \) be sequence for monomial chosen from \( e_{\lambda} \) s.t. \( \text{col}(A) = \mu \) & \( \text{row}(A) = \lambda \).

Cor. \( M_{\lambda, \mu} = M_{\mu, \lambda} \)

Proof. Get bijection between matrices by taking transpose.

Proof. If \( M_{\lambda, \mu} \neq 0 \), then \( \mu \leq \lambda^T \). Also, \( M_{\lambda, \lambda^T} = 1 \).

In particular, \( \{e_{\lambda^T} \mid \lambda \text{ partition} \} \) is basis for \( A \).

Proof. Suppose \( M_{\lambda, \mu} \neq 0 \). Then \( \exists A \ (0,1)\)-matrix w/ \( \text{row}(A) = \lambda \), \( \text{col}(A) = \mu \).

Let \( A' \) be result of left-justifying all 1's in each row

\[ \text{row}(A') = \lambda, \quad \text{col}(A') = \lambda^T. \]

For each \( i \), number of 1's in first\( i \) columns of \( A' \)

\[ \geq \lambda_1^T + \cdots + \lambda_i^T \geq \mu_1 + \cdots + \mu_i \]

\[ A_i \Rightarrow \mu \leq \lambda^T. \]

Also note if \( \mu = \lambda^T \), then 1's must already be left-justified

\[ \Rightarrow \text{only one way}, \quad \text{so } M_{\lambda, \lambda^T} = 1. \]
Lemma. Let \( a_{\lambda, \mu} \) be integers indexed by partitions of size \( n \).

Assume that:
- \( a_{\lambda, \lambda} = 1 \) \( \forall \lambda \)
- \( a_{\lambda, \mu} \to 0 \Rightarrow \mu \leq \lambda \)

For any ordering of \( \lambda \) in \( \mathcal{P}(n) \), \( (a_{\lambda, \mu})_{\mu} \) is invertible (over \( \mathbb{N} \)), i.e. has \( \det = \pm 1 \).

Same conclusion if instead we have \( a_{\lambda, \lambda} = 1 \), \( a_{\lambda, \mu} \to 0 \Rightarrow \mu \leq \lambda \).

\[ \Rightarrow \det (M_{\lambda, \mu}) = \pm 1 \]

Then, \( \{ e_\lambda(x_1, \ldots, x_n) \mid \lambda \leq n \} \) is basis for \( \Lambda(n) \).

Rank. \( \{ e_\lambda \} \) basis for \( \Lambda \Rightarrow e_1, e_2, \ldots \) are algebraically independent. All nontrivial polynomial expressions in \( e_1, e_2, \ldots \) are nonzero.

\( \{ e_\lambda(x_1, \ldots, x_n) \mid \lambda \leq n \} \) basis for \( \Lambda(n) \).

\[ \Rightarrow e_1(x_1, \ldots, x_n), \ldots, e_n(x_1, \ldots, x_n) \text{ are algebraically independent.} \]

Fundamental theorem of symmetric polynomials / functions.