Garnir relations

\[ \lambda = \text{partition}, \; \tau = \lambda - \text{tableau}, \; n = (\lambda) \]

\[ X = \text{some subset of values of boxes in } i\text{th column of } \tau \]

\[ Y = \text{(i+1)st column of } \tau \]

\[ G_X = \text{permutations of } X, \; G_Y = \text{permutations of } Y. \]

\[ G_{X \cup Y} = \text{permutations of } X \cup Y, \; G_X \circ G_Y \subseteq G_{X \cup Y}. \]

Pick coset representatives \( \sigma_1, \ldots, \sigma_k \) for \( G_{X \cup Y} / G_X \times G_Y \)

Define Garnir element \( G_{X,Y} = \sum_{j=1}^{k} \text{sgn}(\sigma_j) \sigma_j \in k[G_n] \)

Ex. \( \tau = \begin{array}{ccc}
1 & 2 \\
3 & 4 \\
5 &
\end{array} \)
\( X = \{3,5\}, \; Y = \{2,4\} \)

Write elements of \( G_{X \cup Y} \) as \( \sigma(3) \sigma(5) \sigma(2) \sigma(4) \)

Choose reps so that \( \sigma(3) < \sigma(5) \) & \( \sigma(2) < \sigma(4) \)

Representatives are: \( 2345, \; 2435, \; 2534, \; 3425, \; 3524, \; 4523 \)

\[ G_{X,Y} \tau = -e_{14} + e_{13} - e_{12} + e_{12} - e_{12} + e_{12} \]
\[ \begin{array}{cccccc}
2 & 5 & 2 & 5 & 2 & 4 \\
5 & 4 & 5 & 9 & 5 & 3 \\
5 & 3 & 4 & 3 & 3 & 5 \\
\end{array} \]

Then \( \text{(Garnir relations)} \) If \( |X \cup Y| > \chi^T \) (size of i\text{th column of } Y(\lambda))

then \( G_{X,Y} \tau = 0. \)

Proof. The left side has integer coefficients in tabloid basis, so to check if it is 0, suffices to assume \( k = \mathbb{Q} \).

Define \[ \alpha = \sum_{\sigma \in G_X \times G_Y} \text{sgn}(\sigma) \cdot \sigma, \; \beta = \sum_{\sigma \in G_{X \cup Y}} \text{sgn}(\sigma) \cdot \sigma \]

Since \( |X \cup Y| > \chi^T \) for every \( \tau \in C_\lambda \), there are always two values from \( X \cup Y \) that are in some row of \( \tau \).
\[ \beta \{ t \} = 0 \quad \text{for any } t \in \mathcal{C}_t \] of \( X \cup Y \)

Why? let \( \pi \) be transposition swapping two elements \( v \) in same row of \( t \). Then for \( \sigma \in \mathcal{G}_{X \cup Y} \), \( \sigma \pi \in \mathcal{G}_{X \cup Y} \) and \( \sigma \{ t \} = \sigma \pi \{ t \} \)

but \( \text{sgn} (\pi) = -\text{sgn} (\sigma \pi) \) and \( \sigma \rightarrow \sigma \pi \) gives bijection between one-half of \( \mathcal{G}_{X \cup Y} \) and other half.

\[ \Rightarrow \beta \{ t \} = \beta K_t \{ t \} = 0 \]

Next, \( \alpha \) is a factor of \( K_t \). Namely, if we pick coset representatives \( \alpha_1, \ldots, \alpha_r \) of \( \mathcal{C}_t / (G_{X \cup Y}) \), then

\[ K_t = (\sum \text{sgn} (\alpha_i) \alpha_i) \alpha \]

Similarly, \( \beta = G_{X \cup Y} \alpha \).

For any \( \sigma \in \mathcal{G}_{X \cup Y} \), we have \( \sigma \cdot K_t = \text{sgn} (\sigma) K_t \).

\[ \Rightarrow 0 = \beta \{ t \} = \beta K_t \{ t \} = G_{X \cup Y} \alpha K_t \{ t \} = G_{X \cup Y} \sum_{\sigma \in \mathcal{G}_{X \cup Y}} \text{sgn} (\sigma) K_t \{ t \} = 1 \cdot x! \cdot y! \cdot G_{X \cup Y} \alpha \]

\[ \Rightarrow G_{X \cup Y} \alpha = 0. \]

**Def.** A tableau \( t \) is standard if values increase left to right in each row \& top to bottom in each column:

\[ t_{ij} < t_{i'j'} \quad \text{\&} \quad t_{ij} < t_{ij'} \]

If \( t \) standard, \( e_t \) is standard polytabloid, \( \{ e_t \} \) standard tabloid.

**Def.** Total ordering on \( \lambda \)-tabloids: \( \{ t_1 \} < \{ t_2 \} \) if \( \exists i, j \) st:

1. For all \( j > i \), \( j \) is in same row of \( \{ t_1 \} \) \& \( \{ t_2 \} \)
2. \( i \) is in higher row of \( \{ t_1 \} \) than in \( \{ t_2 \} \).
\[ \begin{align*} 
345 & \quad 245^2 \quad 145^4 \\
12 & \quad 125 \quad 234 \quad 134^3 \quad 124 \quad 4 \quad 123 \\
\end{align*} \]

**Def.** Given tableau \( t \), let \([t]\) be "column version" of tableau, namely equivalence class under considering two tableaux same if they have same entries in each column.

\([t_1]\prec [t_2]\) same definition if we replace "row" w/ "column" "higher" w/ "more to right".

**Thm.** \( \{ [t] \mid t \text{ standard} \} \) is a basis for \( S^n \).

**Prf.** Linearly independent. If \( t \) standard tableau and \( \sigma \in C_t \), then \( \{ \sigma \} \supset \{ t \} \).

\( t \) id.

Suppose \( t_1, \ldots, t_r \) are distinct standard tableaux and have

\[ C_1 \cdot t_1 + \cdots + C_r \cdot t_r = 0. \]

Assume \( \{ t_1 \} \prec \cdots \prec \{ t_r \} \).

\( \Rightarrow \) \( \{ t_r \} \) appears w/ coeff. \( C_r \).

\( \Rightarrow \) \( C_r = 0 \) since tableaux linearly ind. in \( M_n \).

\( \Rightarrow \) repeat to see that \( C_1 = \cdots = C_{r-1} = 0. \)

**Span:** Need to show for every tableau \( t \) that

\( et \in \text{span} \{ e_S \mid S \text{ standard tableau} \}. \)

Note: if \( [t] = [t'] \), then \( et = \pm et' \).

So we may assume entries in each column of \( t \) increase top to bottom.

We will prove that \( et \in \text{span} \ldots \) by descending induction on column equivalence classes.
Base case: largest equivalence class: put $1, \ldots, \lambda_T$ in first column, $\lambda_T+1, \ldots, \lambda_T+\lambda_1$ in second column, etc.

This is standard, so nothing to show.

Induction step: let $t$ non-standard tableau,

$\exists i,j$ s.t. $t_{ji} > t_{ji+1}$ (row $j$, columns $i, i+1$)

let $X =$ values in column $i$ in rows $j$ and below

$Y =$ values in column $i+1$ in rows $j$ and above.

$|X \cup Y| = \lambda_T + 1$

Pick exact reps $\sigma_1, \ldots, \sigma_k$ for $G_{X \cup Y} / G_X \cup G_Y$.

Assume $\sigma_1 = \text{id}$. We have $G_{X,Y} e_t = 0$

$$e_t = - \sum_{r=2}^{k} \text{sgn}(\sigma_r) e_{\sigma_r t}$$

Claim: $|\sigma_r t| > |t|$ for $r=2, \ldots, k$.

Note: $t_1, i+1 < t_2, i+1 < \ldots < t_{j+1}, i < t_j, i < t_{j+1}, i < \ldots < t_{\lambda_T}, i$

For each $\sigma_r$ ($r \geq 2$), some element in $X$ gets moved one column to the right. Consider the biggest one. This will be the breaker when comparing $|\sigma_r t| > |t|$.

Cor.: dim $S^d = \#$ standard tableaux of shape $A$.

Does not depend on field!