A is noetherian if every submodule of M is finitely generated.
M is noetherian if every submodule of M is finitely generated.
M satisfies ascending chain condition (ACC) if, given any
chain of submodules \( M_1 \subseteq M_2 \subseteq \ldots \) we have \( M_n = M_{n+1} \) for \( n > 0 \).
i.e., the chain stabilizes.

Prop. TFAE: ① M is noetherian
② M satisfies ACC
③ Every nonempty set of submodules of M has a maximal element.

Proof. ①⇒②: Let \( M_1 \subseteq M_2 \subseteq \ldots \) be given. Let \( M' = \bigcup M_n \)
be a submodule of M, hence \( \exists j \) let \( m_j, \ldots m_n \) be generators.
They must belong to some \( M_n \), but then \( M' = M_n = M_{n+1} = \ldots \)
②⇒③: Let \( \mathcal{S} \) be a nonempty set of submodules of M.
Suppose it has no maximal element. We can create infinite
increasing chain of submodules: Pick any \( M_1 \in \mathcal{S} \)
Next, assuming \( M_1 \subseteq \ldots \subseteq M_n \) given, \( M_n \) is not maximal,
so \( \exists M_{n+1} \supsetneq M_n \) also \( \exists J \) continue to get chain.
③⇒①: Suppose M has no f.g. submodule. i.e., ③sequence
\( x_1, x_2, \ldots, x_m \in M \) s.t. \( x_i \) is not in submodule generated by
\( x_1, x_2, \ldots, x_{i-1} \). Let \( M_i = \text{submodule generated by } x_1, \ldots, x_i \).
But then \( S = \bigcap M_1, M_2, \ldots \) has no maximal element.
Prop. Let \( 0 \to M_1 \to M_2 \to M_3 \to 0 \) be short exact sequence of \( A \)-modules. Then \( M_2 \) north \( \iff \) \( M_1 \) \& \( M_3 \) are north.

Proof. Suppose \( M_2 \) is north.

Every chain of submodules of \( M_1 \) is also a chain of submodules of \( M_2 \) hence stabilizes \( \iff \) \( M_1 \) north.

Given chain of submodules in \( M_3 \), \( f \) inverse image in \( M_2 \) is a chain and stabilizes \( \iff \) original chain stabilize \( \iff \) \( M_3 \) north.

Now suppose \( M_1 \) \& \( M_3 \) are north.

Note: Given \( N \subseteq N' \) submodules of \( M_2 \), we have \( N = N' \iff N \cap M_1 = N' \cap M_1 \) \& \( f(N) = f(N') \).

(For any \( x \in N' \setminus N \), either \( x \in M_1 \), or \( x \in N' \setminus N \) violates \( N \cap M_1 \) \& \( f(N) = f(N') \).

\[ N \cap M_1 = N' \cap M_1 \iff f(N) = f(N') \] (violates \( x \in N' \setminus N \)).

Given \( N_1 \subseteq N_2 \subseteq \ldots \subseteq M_2 \), then

\[ N_1 \cap M_1 \subseteq N_2 \cap M_2 \subseteq \ldots \cap f(N_1) \subseteq f(N_2) \subseteq \ldots \text{ stabilizes} \]

\[ \Rightarrow N_1 \subseteq N_2 \subseteq \ldots \text{ stabilizes} \]

Cor. Finite direct sum of north modules is north.

Cor. Consider \( 0 \to M \to M \oplus N \to N \to 0 \).

Cor. If \( A \) is north ring, then every \( f \) \& \( g \)-module is north.

pf. \( M \) is \( f \)- \( g \)- surjection \( A^\oplus n \to M \) for some \( n \).

By previous cor., \( A^\oplus n \) is north, hence so is \( M \).
Noetherian rings

Prop. If $A$ is noeth, so is $A/I$ for any ideal $I$.

Prop. If $A$ is noeth., $S \subseteq A$ multi. subset, Then $S^{-1}A$ is noeth.

pf. Every ideal of $S^{-1}A$ is of the form $S^{-1}I$ for some ideal $I \subseteq A$. If $f \in \neq S^{-1}I$, $fg \in S^{-1}I$ $\forall g$.

\[ I = A \text{ ideal, a minimal prime } p \text{ of } I \text{ is a prime that contains } I \text{ and is minimal (via inclusion) with this property.} \]

Prop. If $A$ is noeth, then set of minimal primes of $I$ is finite.

pf. Suppose not. Then the set of ideals w/ infinitely many minimal primes is nonempty, hence has a maximal element, call it $J$. $J$ not prime, so $\exists x, y \in A$ s.t. $xy \notin J$, but $x \notin J$, $y \notin J$. Let $p$ be min. prime of $J$. Then $xy \in p$ $\Rightarrow x \in p$ or $y \in p$. $\Rightarrow p \supseteq (J, x)$ or $p \supseteq (J, y)$ $\Rightarrow p \text{ min. prime of } (J, x) \text{ or min. prime of } (J, y)$.

By definition, $(J, x)$ & $(J, y)$ have finitely many primes. \[ \square \]

An ideal $I$ is irreducible if: $I = J_1 \cap J_2$ $\Rightarrow I = J_1$ or $I = J_2$ for any ideals $J_1, J_2$.

Prop. Every ideal in noeth. ring can be written as finite intersection of irreducible ideals.
pf. Suppose false, so set of ideals failing this property is nonempty; hence has a maximal element \( I \).

If \( I \) not irreducible \( \implies \exists J, J_2 \supseteq I \) s.t. \( I = J \cap J_2 \)
\( \implies J, J_2 \) are finite intersections of irreducible ideals.
\( \implies \) since we can substitute into \( I = J \cap J_2 \).

Remk. If \( I \) irreducible then \( \text{V}(I) \subseteq \text{Spec} A \) is an irreducible space. Hence every closed subset of \( \text{Spec} A \) can be written as finite union of irreducible closed subsets if \( A \) is noetherian.

Def. Let \( g \subset A \) be proper ideal. Then \( g \) is \underline{primary} if, for all \( x, y \in A \), \( xy \in g \implies x \in g \) or \( y \in g \) for some \( n > 0 \).

i.e., Every zero divisor of \( A/g \) is nilpotent.

Prop. If \( g \) primary, then \( \sqrt{g} \) is prime.

pf. Suppose \( xy \in \sqrt{g} \). Then \( \exists n > 0 \) s.t. \( (xy)^n \in g \).
\( \implies x^n \in g \) (\( \implies x \in \sqrt{g} \)) or \( \exists m > 0 \) s.t. \( (y^m)^n \in g \)
\( \implies (y \in \sqrt{g}) \).

pf. If \( A \) is noetherian, every irreducible ideal \( I \) is primary, in particular, \( \sqrt{I} \) is prime.

If. Prik \( xy \in A/I \) s.t. \( xy = 0 \). Consider chain
\( \text{Ann}(y) \leq \text{Ann}(y^2) \leq \text{Ann}(y^3) \leq \ldots \leq A/I \).
Claim. \((y^n) \cap (x) = 0\).

Pick \(z \in (y^n) \cap (x)_\ast\), so \(a, b \in A/\mathfrak{I}\) s.t.
\[ z = ay^n = bx. \]

\[ \Rightarrow \ yz = bxy = 0 \Rightarrow \ ay^{n+1} = 0 \Rightarrow \ a \in \text{Ann}(y^{n+1}) = \text{Ann}(y^n) \]

and so \(z = ay^n = 0\).

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Let \(x, y \in A\) be representatives for \(x, y \in A/\mathfrak{I}\).

\[ \Rightarrow \ (y^n) + \mathfrak{I} \cap (x) + \mathfrak{I} = \mathfrak{I}. \]

\(I\) irreducible so either \(\tilde{y}^n \in \mathfrak{I} \Rightarrow y\) nilpotent in \(A/\mathfrak{I}\)

or \(\tilde{x} \in \mathfrak{I} \Rightarrow x = 0\)

\[ \Rightarrow \ I \text{ primary}. \]

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Ex. Let \(A = \mathbb{Q}[x, y]/, m = (x, y)\).

Then \(m^2 = (x^2, x, y, y^2)\) is primary, but
\[ (x^2, xy, y^2) = (x, y^2) \cap (x^2, y) \text{ so is reducible}. \]