Inverse Limits

Setup: \( G_0, G_1, G_2, \ldots \) abelian groups

\( \Theta_i : G_n \rightarrow G_{n-1} \) group homomorphism for all \( n \geq 1 \)

Denote it by \( (G_i, \Theta_i) \) or just \( G_i \)

The inverse limit is

\[
\lim_{\rightarrow} G_i = \left\{ (g_0, g_1, \ldots) \in \prod_{i \geq 0} G_i \mid \Theta_i(g_i) = g_{i-1} \quad \forall i \geq 1 \right\}
\]

subgroup of \( \prod_{i \geq 0} G_i \)

If \( G_i \) are rings, \( \Theta_i \) ring homomorphisms, \( \lim_{\rightarrow} G_i \) is also a ring.

\((G_i, \Theta_i), (G'_i, \Theta'_i)\) are inverse systems.

A morphism \( f : (G_i, \Theta_i) \rightarrow (G'_i, \Theta'_i) \) is a sequence of homomorphisms

\[
f_i : G_i \rightarrow G'_i
\]

\( \Theta_i \downarrow \quad \Theta'_i \downarrow \)

\( G_{i-1} \rightarrow G'_{i-1} \)

\( f \) is injective (resp. surjective) if all \( f_i \) have that property

For abelian groups \( \{ A_i \} \xrightarrow{f_i} \{ B_i \} \xrightarrow{g_i} \{ C_i \} \) is exact if

\[
A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i \text{ is exact for all } i.
\]

An inverse system \((A_i, \Theta_i)\) is surjective if \( \Theta_i \) are surjective for all \( i \geq 1 \).

Proof: Given \( 0 \rightarrow \{ A_i \} \rightarrow \{ B_i \} \rightarrow \{ C_i \} \rightarrow 0 \) short exact sequence of inverse systems, get exact sequence

\[
0 \rightarrow \lim_{\leftarrow} A_i \rightarrow \lim_{\leftarrow} B_i \rightarrow \lim_{\leftarrow} C_i
\]

If \( \{ A_i \} \) is surjective, get short exact sequence

\[
0 \rightarrow \lim_{\leftarrow} A_i \rightarrow \lim_{\leftarrow} B_i \rightarrow \lim_{\leftarrow} C_i
\]
Define \( A = \prod_{i \geq 0} A_i \). Define
\[
d^A : A \to A \quad d^A(a_i) = (a_i - \Theta_{i+1}(a_{i+1})).
\]

By definition, \( \ker d^A = \lim_{\leftarrow i} A_i \).

Define \( B, C, d_B, d_C \) similarly. We have short exact sequence on direct products
\[
0 \to A \to B \to C \to 0
\]
\[
d^A \downarrow \quad d^B \downarrow \quad d^C
\]
\[
o \to A \to B \to C \to 0
\]

**Snake Lemma:** get exact sequence
\[
0 \to \lim_{\leftarrow i} A_i \to \lim_{\leftarrow i} B_i \to \lim_{\leftarrow i} C_i \to \text{coker } d^A \to \text{coker } d^B \to \text{coker } d^C \to 0
\]

**Claim:** If \( \{ A_i \} \) is surjective, then \( d^A \) is surjective.

**Pick** \( (a_i) \in A \). Want \( (a_i) \in A_s.i. \), \( d^A(a) = a \).

**Construct** \( a_i \) by induction on \( i \). Set \( a_0 = 0 \).

Given \( a_0, \ldots, a_{n-1} \) s.t. \( a_i - \Theta_{i+1}(a_{i+1}) = a_i \) for \( i = 0, \ldots, n-1 \), let \( a_{n+1} \in A_{n+1} \) be any element s.t. (possible since \( \Theta_{n+1} \) surjective).

\( \Theta_{n+1}(a_{n+1}) = a_n - a_n \Rightarrow a_n - \Theta_{n+1}(a_{n+1}) = a_n \)

**Important Example:** \( P = \text{abelian group} \)
\( P = P_0 \geq P_1 \geq P_2 \geq \ldots \) decreasing sequence of subgroups.

Set \( G_i = P/P_i, \quad \Theta_i : P/P_i \to P/P_{i-1} \)
be natural quotient map (always surjective).

Set \( \hat{P} : = \lim_{\leftarrow i} P/P_i \)

There is map \( 0 \to (g + P_i)_i \).
If \( P \) is any, \( P_i \) are ideals, \( \hat{P} \) is also a ring.

**Ex.** Let \( k \) be a ring. \( \hat{P} = k[[x]] \)

\[ P_i = (x^i). \quad \hat{P}_i = k[[x]]/x^i \text{ is a power series } \]

\( f(x) = \sum_{i \geq 0} a_i x^i, \quad a_i \in k. \iff (a_0 + a_1 x + \cdots + a_i x^i \mod x^i) \)

image of \( P \to \hat{P}_i \) is power series s.t. \( a_i = 0 \) for \( i \gg 0 \).

**Ex.** \( P = \mathbb{Z}, \; p = \text{prime}, \; \hat{P}_i = (p^i), \; \hat{P} = \langle \hat{p} \rangle \) is "ring of \( p \)-adic integers" denoted \( \mathbb{Z}_p \).

By definition, elements of \( \mathbb{Z}_p \) are sequences

\[(a_0, a_1, a_2, \ldots) \text{ s.t. } a_i \in \mathbb{Z}/p^i \text{ and } a_i \equiv a_{i+1} \mod (p^i) \]

Using convention that least representatives for \( \mathbb{Z}/n \) are \( \{0, \ldots, n-1\} \).

Then there exist integers \( b_0, b_1, \ldots \) w/ \( 0 \leq b_i \leq p-1 \) s.t.

\[ a_i = b_0 + b_1 p + \cdots + b_{i-1} p^{i-1} \]

Can use \( b_i \)'s to represent elements of \( \mathbb{Z}_p \) as infinite sums

\[ \sum_{i \geq 0} b_i p^i \quad (0 \leq b_i \leq p-1) \]

**Note:** \(-1 \in \mathbb{Z}_p \) is sum \( \sum_{i \geq 0} (p-1)_i p^i \)

**Note:** \( \mathbb{Z}_p \) is local, w/ maximal ideal generated by \( p \).

\( \overline{Q}_p = \mathbb{Q}_p(\mathbb{Z}_p) = \text{field of } \langle \text{p-adic numbers} \rangle \)

**Cor.** Let \( 0 \to P' \to P \xrightarrow{f} P'' \to 0 \) be a short exact sequence of abelian groups. Given \( P = P_0 \supseteq P_1 \supseteq P_2 \supseteq \cdots \)

define \( \hat{P}' = \hat{P}' \cap \hat{P}_i \), \( \hat{P}'' = f(\hat{P}_i) \).

Then we get short exact sequence

\[ 0 \to \hat{P}' \to \hat{P} \to \hat{P}'' \to 0. \]
In particular, take $P' = P_n$ get identification of $\hat{P}_n$ as subgroup of $\hat{P}$. Get filtration

$$\hat{P} = \hat{P}_0 \supseteq \hat{P}_1 \supseteq \hat{P}_2 \supseteq \cdots$$

Can take inverse limit again: $\hat{P} = \varprojlim \hat{P}_i$.

Prop. The natural map $\hat{P} \to \hat{P}$ is isomorphism.

For each $n$, apply corollary above to short exact sequence

$$0 \to P_n \to P \to P/P_n \to 0$$

Filtration for $P/P_n$ is $P_0/P_n \supseteq P_1/P_n \supseteq P_2/P_n \supseteq \cdots \supseteq P_{n-1}/P_n \supseteq 0$

$$\Rightarrow (P/P_n)_i = (P/P_n)/P_i = P_i/P_n$$ for $i \geq n$

and maps between them are identity.

$\Rightarrow$ the map $P/P_n \to \hat{P}/P_n$ is an isomorphism

$$\Rightarrow \hat{P} \to \varprojlim \hat{P}_i \cong \varprojlim \hat{P}/P_n = \hat{P}$$