Prop. \( A = d \)-dimensional noetherian local ring, \( m = \text{maximal ideal} \).
\( x_1, \ldots, x_d \) system of parameters, \( \mathcal{B} = (x_1, \ldots, x_d) \)
\( \text{let } f \in A[t_1, \ldots, t_d] \) homogeneous of degree \( s \) s.t.
\( f(x_1, \ldots, xd) \in \mathcal{B}^{st+1} \).
\( \text{let } \bar{f} = \text{reduction of module } \mathcal{B} \) Then \( \bar{f} \) is a zerodivisor in
\[ (A/\mathcal{B})[t_1, \ldots, t_d] \]
\( \quad \text{Proof.} \quad \)

Define \( \alpha : (A/\mathcal{B})[t_1, \ldots, t_d] \to \text{gr}_{\mathcal{B}}(A) = \bigoplus_{i \geq 0} A/\mathcal{B}^{st+i} \)
\( \quad \alpha(t_{ji}) = x_i \quad \text{(in } A/\mathcal{B}^2) \)

Since \( f \) homog of deg \( s \), \( \alpha(f) \in \mathcal{B}^s/\mathcal{B}^{st+1} \), so \( \bar{f} \) is not zerodivisor.
\( \Rightarrow \) have surjection \( (A/\mathcal{B})[t_1, \ldots, t_d]/\bar{f} \to \text{gr}_\mathcal{B}(A) \)
\( \Rightarrow \) For all \( n \), have \( \text{dim } ((A/\mathcal{B})[t_1, \ldots, t_d]/\bar{f})_n \geq (\text{gr}_\mathcal{B}(A))_n \)

For \( n \gg 0 \), both sides given by polynomial \( (\star) \)

First, \( n \to \text{dim } ((A/\mathcal{B})[t_1, \ldots, t_d])_n \)

is polynomial of deg \( d-1 \).

Suppose \( \bar{f} \) not zerodivisor. Then have exact sequence
\[ 0 \to (A/\mathcal{B})[t_1, \ldots, t_d] \to (A/\mathcal{B})[t_1, \ldots, t_d]/(\bar{f}) \to (A/\mathcal{B})[t_1, \ldots, t_d]/(\bar{f})/\mathcal{B}^{st+i} \to 0 \]
\( \Rightarrow (\star) \) has degree \( \leq d-2 \)
\( \Rightarrow \bar{f} \) is a zerodivisor.
Cor. Let $A = k$-dim. noetherian local ring, $m = maximal \ ideal$.
Assume $E$ field $k \subset A$. Then any system of parameters in $A$ is algebraically independent over $k$.

Proof. Suppose $x_1, \ldots, x_d$ is system of parameters which is algebraically dependent over $k$. So there exists $f \in k[x_1, \ldots, x_d]$ s.t. $f(x_1, \ldots, x_d) = 0$.

Let $S = \text{smallest degree of nonzero monomial in } f$.

Let $f_s = \text{sum of degree } S \text{ terms in } f$, let $g = f - f_s$.

$\Rightarrow g(x_1, \ldots, x_d) \in \langle x_1, \ldots, x_d \rangle^{S+1}$

$\Rightarrow f_s(x_1, \ldots, x_d) \in \langle x_1, \ldots, x_d \rangle^{S+1}$

$\Rightarrow f_s \in A/(x_1, \ldots, x_d)[[t_1, \ldots, t_d]]$ is a zero divisor.

But, $1k \to A \to A/(x_1, \ldots, x_d)$ is injective, so $f_s$ has coeff. in $k$ and is "same polynomial" as $f_s$.

And any polynomial whose coefficients are units cannot be zero divisor. $\square$

Suppose $k$ field, $A$ is $k$-alg. (which is a domain).

$d_m A = \sup \{ \dim A_m \}$

If $\dim A_m = d$, $\exists f_{1}, \ldots, f_{d} \in \text{Frac}(A)$ which are alg. ind. over $k$.

Recall, given field extension $k \to E$, a transcendence basis for $E$ over $k$ is a maximal set of algebraically independent elements.
1. All transcendence bases have same size. (call it \( \text{transcendence degree} \))
2. If \( E \) is f.g. over \( k \) as a field, then transcendence degree is finite.

**Proof.** Let \( k \) algebraically closed field.

\[ \text{A} = \text{f.g.} k\text{-algebra which is domain.} \]

Then, \( \dim A = \text{transcendence degree of } \text{Frac}(A) \mid k. \)

Furthermore, \( \dim A = \dim A_m \) for all maximal ideals \( m. \)

**If.** First suppose \( A = k[x_1, \ldots, x_d] \) polynomial ring.

\[ \text{Frac}(A) = k(x_1, \ldots, x_d) \] has transcendence degree \( d \) over \( k. \)

By Nullstellensatz, every maximal ideal is of the form \( (x_1 - \alpha_1, \ldots, x_d - \alpha_d) \) for some \( \alpha \in \mathbb{C} \). So they're all related by linear change of coordinates.

\[ \implies \dim A_m \text{ independent of } m. \]

\[ \mathfrak{m} (x_1, \ldots, x_d) \cong A \implies \dim A = d. \quad \checkmark \]

**General case:** Noether normalization gives:

\[ B \subset A \text{ s.t. } A \text{ is integral over } B \text{ and } B = k[x_1, \ldots, x_d] \]

polynomial ring. \( \implies \text{Frac}(A) \text{ algebraic over } \text{Frac}(B), \)

so both have same transcendence degree over \( k. \) (\( = d \))

let \( m \subset A \) be maximal ideal.

Then \( m = \mathfrak{m} \cap B \) is also maximal.

Let \( P_0 \neq \cdots \neq P_r \) be chain of prime ideals in \( A_m. \)
Intersect w/ $B_n$: \[ p_0 \cap B_n \subseteq \cdots \subseteq p_0 \cap B_n. \]

\[ \implies d = \dim B_n \geq \dim A_m. \]

On the other hand, any strict chain of prime ideals in $B_n$ comes from intersecting strict chains of prime ideals in $A_n$ by "going-down thru" \( \Rightarrow \dim A_m \geq \dim B_n. \)

\[ \Rightarrow \dim A_m = d \quad \forall \text{ maximal ideals } m. \]

\[ \Rightarrow \dim A = d. \]