Hilbert–Samuel polynomial

\[ A = \text{n.c. local ring, } m = \text{maximal ideal} \]
\[ g = m\text{-primary ideal (i.e., } \sqrt{g} = m) \text{ generated by } x_1, \ldots, x_s \in A \]

Prop. \( M \) is finitely gen. \( A \)-module, \( \mathcal{F} \) stable \( g \)-filtration of \( M \). Then:

1. \( M/M_n \) is finite length module for all \( n \geq 0 \).
2. \( g \) polynomial \( g(x) \) of degree \( \leq S \) s.t. \( g(x) = l(M/M_n) \) for \( n \geq 0 \).

Furthermore, \( \text{deg}(g(x)) = \text{order of pole at } t = 1 \text{ of } H_{\mathcal{F} \text{-gr}}(M)(t) \)

3. The degree and leading coeff. of \( g(x) \) do not depend on choice of stable \( g \)-filtration. (Only on \( M \) and \( g \)).

Pf. Set \( B := \text{gr}_g(A) = \bigoplus_{n \geq 0} g^n/g^{n+1} \)

\[ N := \text{gr}_g(M) = \bigoplus_{n \geq 0} M_n/M_{n+1} \]

\( B \) is n.c. ring, \( N \) is finitely gen. \( B \)-module.

\[ N_n = M_n/M_{n+1} \text{ is a } \text{f.g. } B_0 \text{-module for all } n. \]

Since \( \sqrt{g} = m \), \( A/g \) is artinian \( \Rightarrow M_n/M_{n+1} \) is finite length for all \( n \).

We have exact sequences:

\[ 0 \rightarrow M_{n-1}/M_n \rightarrow M/M_n \rightarrow M/M_{n+1} \rightarrow 0 \]

\[ \Rightarrow l(M/M_n) = \sum_{i=0}^{n-1} l(M_i/M_{i+1}) < \infty \]

\[ \Rightarrow (1-t) \sum_{n \geq 0} l(M/M_n) t^n = t \sum_{n \geq 0} l(M_n/M_{n+1}) t^n \]
\( B \) is generated by degree 1 elements: the images of \( x_1, \ldots, x_s \) under \( y \to y^{q^2} \).

\[ H_N(t) = \sum_{n \geq 0} \left( \frac{\ell(M/M_n)}{m} \right)^n t^n \text{ is a rational function in } t, \]

with denominator \( (1-t)^5 \).

\[ \Rightarrow \sum_{n \geq 0} \left( \frac{\ell(M/M_n)}{m} \right)^n = \frac{\text{polynomial}(t)}{(1-t)^{5s+1}}. \]

\[ \Rightarrow \exists \text{ a polynomial } g(x) \text{ of degree } \leq s \text{ s.t. } g(n) = \ell(M/M_n) \text{ for } n \geq 0. \]

\[ \exists \text{ a polynomial } g'(x) \text{ of degree } \leq s \text{ s.t. } g'(n) = \ell(M/M'_n) \text{ for } n \geq 0. \]

\[ \exists \text{ a polynomial } g(x) \text{ of degree } \leq s \text{ s.t. } g(n) = \ell(M/M_n) \text{ for } n \geq 0. \]

There exists \( M_{n+1} \subseteq M_n \) for all \( n \geq 0 \).

\[ \Rightarrow \ell(M/M_{n+1}) = \ell(M/M_n) \text{ for all } n \geq 0, \]

\[ \ell(M/M_{n+1}) = \ell(M/M_n) \]

\[ \Rightarrow \text{ for } n \geq 0, g(n+1) = g(n) \geq g(n-n_0) \]

Divide by \( g(n) \):

\[ \Rightarrow \frac{g(n)}{g(n-n_0)} = \frac{g(n-n_0)}{g(n)} \text{ for } n \geq 0. \]

As \( n \to \infty \) we get \( \ell = \lim_{n \to \infty} \frac{g(n)}{g(n-n_0)} \).

\[ \Rightarrow g'(x) \text{ and } g(x) \text{ have same leading coeff. and degree}. \]

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The Hilbert-Samuel polynomial of \( M \) (with respect to \( g \)) is denoted \( \chi^M_g(x) \), and is unique polynomial s.t.
\[ x_q^M(n) = l\left( M/q^n M \right) \quad \text{for } n \gg 0. \]

If \( M = A \), write \( x_q^A(x) \) for \( x_q^M(x) \).

Prop. \( \deg x_q^A(x) = \deg x_m(x) \).

If \( \dim A \cdot q = m \Rightarrow \exists \text{ st. } m^d \leq q \)

For all \( n \gg 0 \), \( m^d \leq q^n \leq m^n \)

\[ \Rightarrow l\left( A/m^d \right) \geq l\left( A/q^n \right) \geq l\left( A/m^n \right) \]

\[ \Rightarrow \text{ for } n \gg 0 \quad x_m(dn) \geq x_q(n) \geq x_m(n) \]

\[ \Rightarrow \deg x_m(x) = \deg x_q(x). \]

Summary: (Set \( k = A/m \))

\[ (1-t) \sum_{n \geq 0} \dim_k \left( A/m^n \right) t^n = t \sum_{n \geq 0} \dim_k \left( m^d/m^{n+1} \right) t^n \]

\[ = t \cdot H_{grm(A)}(t) \]

\[ \Rightarrow \deg x_m(x) = \text{order of pole at } t=1 \text{ of } H_{grm(A)}(t) \]

Prop. \( x \in A \) a nonzerodivisor on \( M \).

Set \( M' = M/xM \). Then: \( \deg x_q^{M'} \leq \deg x_q^M - 1 \).

Prop. Set \( N = xM \), \( N_0 = N \cap q^n M \)

(By Artin-Rees Lemma, \( N_0 \) gives suitable \( q \)-filtration on \( N \))

Exact sequence: \( 0 \rightarrow N/N_0 \rightarrow N/q^n M \rightarrow M'/q^n M' \rightarrow 0 \)
for $n \gg 0$, $\chi_{q'}^M(n) = \chi_q^M(n) - (dL(N/N_n))$

Since $x$ is NBD on $M$, the map $M \to N$ is an isomorphism. For $n \gg 0$, $dL(N/N_n)$ is a polynomial function which has same degree and leading coeff as $\chi_{q'}^N(x) = \chi_q^M(x)$

$\Rightarrow \deg \chi_{q'}^M(x) \leq \deg \chi_q^M(x) - 1$. \qed