Math 190A, Fall 2022
Homework 7
This is for practice only, don’t turn it in. But there’s a very high chance that one of these problems will appear on the final exam.

(1) Let $X$ be a compact metrizable space. Prove that $X$ is second-countable.

(2) Let $X$ be a regular space. Prove that for all $x, y \in X$ with $x \neq y$, there exist neighborhoods $U$ and $V$ of $x$ and $y$, respectively, such that $\overline{U} \cap \overline{V} = \emptyset$.

(3) A space $X$ is called **completely regular** if it is a $T_1$-space, and given any closed subset $A$ and $x \notin A$, there exists a continuous function $f: X \to [0,1]$ such that $f(x) = 0$ and $f(a) = 1$ for all $a \in A$.
   (a) Let $X$ be a locally compact and Hausdorff space. Prove that $X$ is completely regular. [Hint: Use $X^\ast$.]
   (b) Prove that completely regular implies regular.
   (c) Use the previous two parts to conclude that every manifold is metrizable.

(4) Let $X$ be a compact and Hausdorff space and assume that for each $x \in X$, there is a neighborhood $U$ of $x$ and a positive integer $k$ such that $U$ has an embedding into $\mathbb{R}^k$. Prove that $X$ has an embedding into $\mathbb{R}^N$ for some $N$.
   [Note: this is close to the definition of locally Euclidean, but we’re allowing $k$ to depend on $x$, and not fixing it ahead of time.]