Math 190A, Fall 2022
Homework 6
Due: Friday, December 2, 2022 11:59PM via Gradescope
(late submissions allowed up until December 3, 2022 11:59PM with −25% penalty)

Solutions must be clearly presented. Incoherent or unclear solutions will lose points.

(1) Let \( X \) and \( Y \) be spaces and set \( \{ p \} = X^* \setminus X \) and \( \{ q \} = Y^* \setminus Y \). Prove that
\[
(\mathbb{R} \coprod Y^*)^* \cong (X^* \coprod Y^*)/\sim
\]
where the only nontrivial relation is \( p \sim q \).

[Recall that \( \coprod \) means “disjoint union”, and if \( A \) and \( B \) are spaces, then a subset \( U \) of \( A \coprod B \) is open if and only if \( U \cap A \) and \( U \cap B \) are open in \( A \) and \( B \), respectively.]

(2) Let \( X = \mathbb{Z}_{>0} \) be the set of positive integers with the discrete topology.
(a) Prove that \( X \) is locally compact, Hausdorff, and not compact.
(b) Prove that \( X^* \) is homeomorphic to the subspace \( \{0\} \cup \{1/d \mid d \in \mathbb{Z}_{>0} \} \) of \( \mathbb{R} \).

(3) Let \( Y \) be a Hausdorff space and let \( X \subseteq Y \) be a locally compact subspace such that \( X = Y \). Prove that \( X \) is an open subset of \( Y \). Hints at end.

(4) Let \( n \geq 1 \) be an integer. Recall that \( \mathbb{RP}^n = (\mathbb{R}^{n+1} \setminus \{0\})/\sim \) where \( x \sim y \) if there exists \( \lambda \in \mathbb{R} \setminus \{0\} \) such that \( x = \lambda y \). Write \([x_1 : \cdots : x_{n+1}]\) for the equivalence class of \((x_1, \ldots, x_{n+1})\). Our goal is to show that \( \mathbb{RP}^n \) is a compactification of \( \mathbb{R}^n \); since it’s a bit lengthy this problem will be worth 40 points rather than the usual 20.
Define \( U \subseteq \mathbb{RP}^n \) to be the subset of equivalence classes of the form \([x_1 : \cdots : x_{n+1}]\) where \( x_{n+1} \neq 0 \) (this makes sense since whether or not \( x_{n+1} \) is 0 does not depend on the actual representative).
(a) Prove that the function \( g: U \to \mathbb{R}^n \) given by
\[
g([x_1 : \cdots : x_{n+1}]) = \left( \frac{x_1}{x_{n+1}}, \ldots, \frac{x_n}{x_{n+1}} \right)
\]
is well-defined (i.e., does not depend on the choice of representative for the equivalence class) and is a homeomorphism. Hint at end.
(b) Prove that \( U = \mathbb{RP}^n \).
(c) Finally, prove that \( \mathbb{RP}^n \) is Hausdorff.
[You may use that the restriction \( \pi|_{S^n}: S^n \to \mathbb{RP}^n \) is a quotient map, i.e., \( U \subseteq \mathbb{RP}^n \) is open if and only if \( (\pi|_{S^n})^{-1}(U) \) is open. This does not follow from definitions and requires a proof, but you can take it for granted for this problem.]
(d) When \( n = 1 \), explain why \( \mathbb{RP}^1 \setminus U \) is a single point and explain how this implies that \( \mathbb{RP}^1 \cong S^1 \).
Hints

3: Hint 1: By Proposition 4.3.19 (taking $U = X$), each $x \in X$ has a neighborhood $V \subseteq X$ which is open in $X$ such that $\text{Cl}_X(V)$ is compact. Prove that $V$ is also open in $Y$ (see next hint for more help).

Hint 2: Continuing from hint 1, $V = X \cap W$ for some open set $W$ in $Y$. Explain why each of the following equalities holds:

$$W \subseteq \text{Cl}_Y(W) = \text{Cl}_Y(V) = \text{Cl}_X(V) \subseteq X.$$  

4a: To show that $g$ is continuous: let $\widetilde{U} = \pi^{-1}(U)$ where $\pi: \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}P^n$ is the quotient map. Define $f: \widetilde{U} \to \mathbb{R}^n$ by the same formula as $g$ and show that $f = (\pi|_{\widetilde{U}}) \circ g$. Now use an argument very similar to the proof of Proposition 2.4.5.

Optional problems (don’t turn in)

(5) How do you describe $(X \times Y)^*$ in terms of $X^*$ and $Y^*$?

(6) Prove that $\mathbb{C}P^n$ is a compactification of $\mathbb{C}^n$ and that $\mathbb{C}P^1 \cong S^2$.

(7) Pick $0 < k < n$. Prove that $\text{Gr}_k(\mathbb{R}^n)$ is a compactification of $\mathbb{R}^n$ and that $\text{Gr}_k(\mathbb{C}^n)$ is a compactification of $\mathbb{C}^n$. 