Choice Problems
\[ [n] = \{1, \ldots, n\} \]. Count subsets of \([n]\) of size \(k\).

Consider the expansion of \((1+x)^n = (1+x)(1+x) \cdots (1+x)\).

Choice of 1 or \(x\) at each step \(\iff\) subset of \([n]\) of size \(k\) where \(x\) is chosen \(k\) times.

\[ n=5 \implies (1+x)(1+x)(1+x)(1+x)(1+x) \iff \{1, 4, 5\} \subseteq [5] \]

Prop. \# subsets of \([n]\) of size \(k\) = \([x^k] (1+x)^n = \binom{n}{k}\)

Pascal's identity: \([x^k] (1+x)^n = [x^k] (1+x)^{n-1} (1+x)\)

\[ \binom{n}{k} \text{ Pascal } = \binom{n-1}{k-1} + \binom{n-1}{k} \]

Our binomial thm: \((1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k\)

Homogenize: \(x \rightarrow \frac{x}{y}\) Multiply by \(y^n\):

\[ (x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \]

Cor. \# subsets of \([n]\) = \(2^n\)

PF. \# subsets of \([n]\) = \(\sum_{k=0}^{n} \# \text{ subsets of size } k \text{ of } [n]\)

\[ = \sum_{k=0}^{n} [x^k] (1+x)^n = \text{ sum of coefficients of } (1+x)^n \]

Plug in \(x=1\): \(2^n\)

Ex. Substitute \(x=2, y=3\) into binomial thm: \(S^n = \sum_{k=0}^{n} \binom{n}{k} 2^k 3^{n-k}\).

Take derivative: \(n(1+x)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} k x^{k-1}\)

\(x \rightarrow 1\): \(n 2^{n-1} = \sum_{k=0}^{n} k \binom{n}{k}\)

\(\square\)
Given \( k_1 + \ldots + k_d = n \) (\( k_i \geq 0 \) integers), define

\[
\text{multinomial coefficient} \quad \binom{n}{k_1, \ldots, k_d} = \frac{n!}{k_1! \cdot k_2! \cdot \ldots \cdot k_d!}
\]

**Thm (Multinomial Thm)**

\[
(x_1 \ldots + x_d)^n = \sum \binom{n}{k_1, \ldots, k_d} x_1^{k_1} \ldots x_d^{k_d}
\]

Sum over all triples \( (k_1, \ldots, k_d) \) s.t. \( k_i \geq 0 \) integers & \( k_1 + \ldots + k_d = n \)

**Pf.** Induction on \( d \). If \( d = 1 \), both sides are \( x_1^n \) \( \checkmark \)

Now assume \( d > 1 \), multinomial thm holds for \( d-1 \) variables

Substitute \( x \rightarrow x_1 + \ldots + x_{d-1} \), \( y \rightarrow x_d \) into binomial thm:

\[
(x_1 + \ldots + x_d)^n = \sum_{m=0}^{n} \binom{n}{m} (x_1 + \ldots + x_{d-1})^m x_d
\]

\[
= \sum_{m=0}^{n} \binom{n}{m} \sum_{k_1 + \ldots + k_{d-1} = m} \binom{m}{k_1, \ldots, k_{d-1}} x_1^{k_1} \ldots x_{d-1}^{k_{d-1}} x_d^{n-m}
\]

Set \( k_d = n - m \); note \( \binom{n}{k_d} \binom{n-k_d}{k_1, \ldots, k_{d-1}} = \frac{n!}{(n-k_d)! k_1! \cdots k_d!} \)

\[
\sum_{k_1 + \ldots + k_{d-1} = n} \binom{n}{k_1, \ldots, k_d} x_1^{k_1} \ldots x_d^{k_d}
\]

What do multinomial coeffs count?

Suppose \( 1, \ldots, d \) represent colors. We have \( n \) objects in a row, and we need to assign a color to each.

\[
(x_1 + \ldots + x_d)^n = (x_1 + \ldots + x_d) \cdots (x_1 + \ldots + x_d)
\]

\( \exists n=3 \)

\[
(x_1 + x_2 + x_3)^3 = (x_1 + x_2 + x_3)(x_1 + x_2 + x_3)(x_1 + x_2 + x_3)
\]

\( \Rightarrow \) color by first object \( 1 \)

last two objects \( 3 \)

\[
\sum x_1 x_2^2
\]
Prop. Assume we have \( d \) types of objects (colors).

Then \( \binom{n}{k_1, \ldots, k_d} \) is the ways to arrange \( n \) objects such that exactly \( k_i \) many objects are of the \( i \)th type.

Ex. 10 houses in a row. We have to paint 4 blue, 2 red, 3 green, 1 orange. Ways to assign colors is \( \binom{10}{4,2,3,1} = 10! / (4! 2! 3! 1!) = 12600 \).

A variation: multisets

A multiset of size \( k \) of \( [n] \) is a choice of \( k \) elements, but we can choose elements more than once.

Ex. \( \{1,1,1,2,2,3,5\} \) is a multiset of size 7 of \( [5] \).

A multiset of size \( k \) of \( [n] \) \( \iff \) a way to multiply out term

\[
\left( 1 + x + x^2 + \cdots \right)^n = \sum_{d=0}^{\infty} x^d \binom{n}{d}^n
\]

\[
1112235 \iff (1 + x + x^2 + x^3 + \cdots)(1 + x + x^2 + \cdots)(1 + x + x^2 + \cdots)(1 + x + x^2 + \cdots)(1 + x + x^2 + \cdots)
\]

\[
\Rightarrow x^7
\]

Note: \( \sum_{d \geq 0} x^d = (1-x)^{-1} \)

Prop. Multisets of size \( k \) of \( [n] \) = \( \sum_{d \geq 0} x^k \binom{n}{d} \binom{n-1}{d} \)

Suggests: We can find bijection

\[
\left\{ \text{multisets of size } k \text{ of } [n] \right\} \leftrightarrow \left\{ \text{subsets of size } k \text{ of } [n+k-1] \right\}
\]