Transfer Matrix Method

Let $A$ be an $n \times n$ square matrix. Define

$$ F_{A; i, j}(x) = \sum_{k=0} \left( A^k \right)_{i,j} x^k $$

Notation: $B = A_{i,j}$, $(B^k)_{i,j} = (n \times 1) \times (n \times 1)$ submatrix obtained by deleting row $j$ & column $i$.

**Thm.**

$$ F_{A; i, j}(x) = (-1)^{i+j} \frac{\det((\text{id}_n - xA)_{i,j})}{\det(\text{id}_n - xA)} $$

so $F_{A; i, j}(x)$ is a rational generating function.

**Prf.** Consider $\sum A^k x^k$ as an $n \times n$ matrix whose entries are formal power series. The inverse of this matrix is $\text{id}_n - xA$. So $F_{A; i, j}(x)$ is the $(i,j)$ entry of $(\text{id}_n - xA)^{-1}$.

Use Cramer's rule to get desired formula.

Let $\lambda_1, \ldots, \lambda_n$ be eigenvalues of $A$. Then

$$ \det(t \cdot \text{id}_n - A) = (t - \lambda_1)(t - \lambda_2) \cdots (t - \lambda_n) $$

$t \to \frac{1}{x}$, multiply by $x^n$:

$$ \det(\text{id}_n - xA) = (1 - \lambda_1 x)(1 - \lambda_2 x) \cdots (1 - \lambda_n x) $$

$\Rightarrow$ $(A^k)_{i,j}$ can be expressed as linear comb. of powers of $\lambda_i^k$ (for $k \in \mathbb{N}$)
Consider length $n$ words in $C^3$ s.t. 11 and 23 never appear in consecutive places.

\[
G = \begin{cases} 1 \\ \downarrow \\ 2 \leftarrow 3 \\ \uparrow \end{cases}
\]

Claim: words of length $n$ walks of length $n-1$ in $G$

Consider words starting w/ 1 and ending at 3.

\[
\det \left( (id_3 - xA; 3, 1) \right) = \det \begin{bmatrix} 1 & -x & -x \\ -x & 1 & 0 \\ -x & -x & 1 \end{bmatrix} = x(1-x) = x-x^2
\]

\[
\Rightarrow F_{A; 1,3}(x) = \frac{x-x^2}{1-2x-x^2+x^3}
\]

\[
\left[x^{n-1}\right] F_{A; 1,3}(x) = \# \text{words} \quad \square
\]

Ex. Tile $n \times k$ chessboard by dominoes of size 1x2 or 2x1.

such as

\[
\begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & & \cdot \\
\end{array}
\]

\[
f_n(k) = \# \text{ ways to do this.}
\]

1. If $n=1$, $f_1(k) = \begin{cases} 1 & \text{if } k \text{ even} \\ 0 & \text{if } k \text{ odd} \end{cases}$

2. If $n=2$, consider rightmost column of a tiling.

   Two possibilities: either has 1 vertical domino
   or 2 horizontal dominoes occupying last 2 columns

\[
\Rightarrow f_2(k) = f_2(k-1) + f_2(k-2), \quad f(1)=1, \quad f(2)=2
\]

\[
\Rightarrow \text{Fibonacci numbers.}
\]
If \( n \leq 3 \), recurrence relation possible, but maybe messy.

General approach for any \( n \): encode tilings as walks in graph \( G_n \).

Encode a tiling by marking the squares which are left square of a horizontal domino. Each column becomes word in \( \{0,1,2\} \) of length \( n \): 1 means marked square, 0 means not marked.

Vertices of \( G_n \) are words of length \( n \) in \( \{0,1,2\} \).

Given words \( w, \bar{w} \), draw edge \( w \rightarrow \bar{w} \) if there is a way to place dominos so that \( \bar{w} \) appears right before \( w \).

Eg. \( n = 2 \):

\[
\begin{align*}
G_2: \quad 10 & \rightarrow 01 \\
11 & \rightarrow 00
\end{align*}
\]

Boundary conditions: rightmost column must be \( 00 \ldots 0 \), leftmost column has restrictions ...

\( \Rightarrow \) \( f_n(k) \) = \#walks of length \( k-1 \) starting at valid leftmost column, ending at \( 00 \ldots 0 \).

\( \sum_{k=0}^{\infty} f_n(k) x^k = \text{finite sum of } F_{A; 1; 0}(x) \) hence rational

\( \Rightarrow \) \( f_n(k) \) satisfies linear recurrence relation
Words that can be interpreted as walks in a graph = "regular languages" are recognized by deterministic finite-state automata.

Ex. Consider words in \( \{0,1\}^* \) such that 1 appears at most twice in a row.

\[
\begin{align*}
0 & \rightarrow 00 \\
0 & \rightarrow 01 \\
1 & \rightarrow 10 \\
1 & \rightarrow 11
\end{align*}
\]

Number of words of length \( n \) = walks of length \( n+1 \) in this graph starting from 0 or 1.