## 12-fold way, summary

<table>
<thead>
<tr>
<th>$k$ balls/boxes</th>
<th>$f$ arbitrary</th>
<th>$f$ injective</th>
<th>$f$ surjective</th>
</tr>
</thead>
<tbody>
<tr>
<td>dist/dist</td>
<td>words $n^k$</td>
<td>injective words $\binom{n}{k}$</td>
<td>ordered set partitions $n! S(k,n)$</td>
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<tr>
<td>indist/dist</td>
<td>weak compositions $\binom{n+k-1}{k}$</td>
<td>subsets $\binom{n}{k}$</td>
<td>compositions $\binom{k-1}{n-1}$</td>
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<tr>
<td>dist/indist</td>
<td>set partitions of $[k]$ w/ $\leq n$ blocks $\sum_{i=1}^{n} S(k,i)$</td>
<td></td>
<td>set partitions of $[k]$ w/ $n$ blocks $S(k,n)$</td>
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<tr>
<td>indist/indist</td>
<td>integer partitions of $k$ w/ $\leq n$ parts $P \leq n(k)$</td>
<td></td>
<td>integer partitions of $k$ w/ $n$ parts $P_n(k)$</td>
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</table>

### Cycles in permutations

$$(x)_k = x(x-1)(x-2)\ldots(x-k+1)$$

**Prop.** $x^n = \sum_{k=0}^{n} S(n,k) (x)_k$

**PF.** Pick positive integer $d \geq n$.

$d^n = \# \text{ functions } [n] \to [d]$

Count such functions by their image $S \subseteq [d]$

$= \# \text{ surjective functions } [n] \to S = 1S_1! S(n,1S_1) = \sum_{S \subseteq [d]} d^n = \sum_{S \subseteq [d]} 1S_1! S(n,1S_1) = \sum_{k=0}^{d} k! S(n,k) \binom{d}{k} = \sum_{k=0}^{d} S(n,k) (d)_k$

**If** $k > n$, then $S(n,k) = 0$

$$\sum_{k=0}^{n} S(n,k) (d)_k$$

$\Rightarrow x^n - \sum_{k=0}^{n} S(n,k) (x)_k$ has infinitely many roots
Nonzero polynomials only have finitely many roots

\[ x^n = \sum_{k=0}^{n} \binom{n}{k} x^k \]

How to write \( (x)_k \) as sum of \( x^n \)?

Cycle decomposition of permutation \( \sigma \in S_n \):

- given \( 1 \leq i \leq n \), consider sequence \( i, \sigma(i), \sigma^2(i), \ldots, \sigma^{k-1}(i) \)

where \( \sigma^k(i) = i \)

Denote \( i \rightarrow \sigma(i) \rightarrow \sigma^2(i) \rightarrow \ldots \rightarrow \sigma^{k-1}(i) \rightarrow i \)

Length \( 1 \) cycles are possible.

Example: \( \sigma = 135624 \)

\[ 1 \rightarrow 1 \]
\[ 2 \rightarrow 3 \rightarrow 5 \rightarrow 2 \]
\[ 4 \rightarrow 6 \rightarrow 4 \]

3 cycles

\( C(n,k) = \# \) permutations \( \sigma \in S_n \) w/ \( k \) cycles

\( C(0,0) = 1 \).

Note: \( C(n,0) = 0 \) if \( n > 0 \)

Prop. \( n \geq k \geq 1: \ C(n,k) = C(n-1,k-1) + (n-1) \cdot C(n-1,k) \)

Pf. Consider 2 types of permutations w/ \( k \) cycles.

Type I. Permutations where \( n \) is its own cycle.

Delete \( n \), get permutation in \( S_{n-1} \) w/ \( k-1 \) cycles.

Can recover permutation uniquely, so get bijection.

There are \( C(n-1,k-1) \) many permutations.
Type II. Permutations where \( n \) is not in its own cycle.

... \( i \to n \to j \to ... \) \( i \neq n \) \( j \neq n \)

Let \( \sigma \) be such a permutation.

Define \( T \in S_{n-1} \) by \( T(i) = j \)

\( T(x) = \sigma(x) \) for all \( x \neq i \).

\( T \) has \( k \) cycles, \( \sigma \) can be recovered if I remember \( i \).

Get bijection \( \{ \text{type II permutations} \} \leftrightarrow \{ \text{pairs } (i, j) \} \)

\# type II = \((n-1) \cdot c(n-1, k)\)

\[ \Rightarrow \quad c(n, k) = c(n-1, k-1) + (n-1) \cdot c(n-1, k) \]

\[ \square \]

Cor. For \( n \geq 0 \),

\[ \sum_{k=0}^{n} c(n, k) x^k = x(x+1) \cdots (x+n-1) \]

where the right side is 1 if \( n = 0 \). In particular,

\[ \sum_{k=0}^{n} (-1)^{n-k} c(n, k) x^k = (x)_n \]

Ref. Prove by induction on \( n \). If \( n = 0 \), get 1 = 1

Now suppose \( n > 0 \). Then \( c(n, 0) = 0 \). We get

\[ \sum_{k=1}^{n} c(n, k) x^k = x \sum_{k=1}^{n} c(n-1, k-1) x^{k-1} + (n-1) \sum_{k=1}^{n} c(n-1, k) x^k \]

\[ = x \sum_{j=1}^{n-1} c(n-1, j) x^j + (n-1) \sum_{j=1}^{n-1} c(n-1, j) x^j \]

\[ = (x + n-1) \sum_{j=1}^{n-1} c(n-1, j) x^j = (x+n-1) \cdot x(x+1) \cdots (x+n-2) \]
Substitute $x \rightarrow -x$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} x^k = (-x)(-x+1) \cdots (-x+n-1)$$

Multiply by $(-1)^n$:

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} x^k = x(x-1) \cdots (x-n+1) = (x)_n \quad \square$$

**Def.** $s(n,k) = (-1)^{n-k} \binom{n}{k}$ Stirling numbers of first kind.

**Cor.** ("inversion formula") For $n, \ell \geq 0$, we have:

$$\sum_{k=0}^{n} S(n,k)s(k,\ell) = \delta_{n,\ell} = \sum_{k=0}^{n} s(n,k)S(k,\ell)$$

**Prf.**

$$x^n = \sum_{k=0}^{n} S(n,k)(x)_k = \sum_{k=0}^{n} S(n,k) \sum_{\ell=0}^{k} s(k,\ell) x^\ell$$

Take coefficient of $x^\ell$:

**Left side:** $\delta_{n,\ell}$

**Right side:** $\sum_{k=0}^{n} S(n,k)s(k,\ell)$

**Second identity:** $(x)_n = \sum_{k=0}^{n} s(n,k) x^k = \sum_{k=0}^{n} s(n,k) \sum_{\ell=0}^{k} S(k,\ell) (x)_\ell$

$(x)_n$ form basis for polynomials as $n$ varies.

Ask for coefficient of $(x)_\ell$ of both sides

**Left side:** $\delta_{n,\ell}$, **right side:** $\sum_{k=0}^{n} s(n,k)S(k,\ell)$.

$\square$