Set partitions

Def. \( X = \text{set} \), partition of \( X \) is an unordered collection of nonempty subsets \( S_1, \ldots, S_k \) s.t. every element of \( X \) belongs to exactly one \( S_i \).

An ordered partition is same, except the \( S_1, \ldots, S_k \) are ordered.

**12-fold:** \( X = \) set of distinguishable balls, \( f \) surjective

\[ k \text{ boxes (distinguishable in ordered case)} \]
\[ \text{indistinguishable else} \]

\( S_i = \) records ball contained in box \( #_i \)

Ex. \( X = \{1,2,3\} \). There are 5 partitions of \( X \):

\[
\begin{align*}
\{1,2,3\}, \{1,2\}, \{3\} & \quad \{1,3\}, \{2\}, \{1\} \\
\{1\}, \{1,2,3\} & \quad \{1\}, \{1,2\}, \{3\} \\
\{1,3\}, \{2\} & \quad \{1,2\}, \{3\}
\end{align*}
\]

ordered partitions = \( 1 + 2 \cdot 3 + 6 = 13 \)

Alternative notation

\[
123 \quad 12|3 \quad 13|2 \quad 23|1 \quad 1|2|3
\]

Ex. 20 pieces of candy (all different)

4 children (clones of each other)

\( \{\text{partitions of candy}\} \leftrightarrow \{\text{assignments of candy to children, so that no one is empty-handed}\} \)

Def. \( S(n,k) = \) \# of partitions of set of size \( n \) into \( k \) blocks

"Stirling number of 2nd kind".

\( S(0,0) = 1 \). \( \underline{Note:} S(n,k) = 0 \) if \( k > n \) or if \( k = 0 \) and \( n > 0 \)

\( \# \text{ ordered partitions of } \{n\} \text{ into } k \text{ blocks} = k! \cdot S(n,k) \)
Ex. \( n \geq 1 \)  \( S(n,1) = 1 \)
\[
S(n,n) = 1
\]
\( n \geq 2 \)  \( S(n,2) = \frac{2^n - 2}{2} = 2^{n-1} - 1 \)

\( S(n,n-1) = \binom{n}{2} \)

Thm. \( n \geq k \geq 1 \)  \( S(n,k) = S(n-1,k-1) + kS(n-1,k) \)

pf. Two types of partitions of \( \{1, \ldots, n\} \):

Type I: Partitions sit, \( n \) is in its own block.
Remove this block: get partition of \( \{1, \ldots, n-1\} \) into \( k-1 \) blocks.
\( \# \text{ Type I} = S(n-1,k-1) \)

Type II: Partitions sit, \( n \) shares its block.
Remove \( n \): get partition of \( \{1, \ldots, n-1\} \) into \( k \) blocks.
\( \# \text{ Type II} = kS(n-1,k) \)

\[\Rightarrow S(n,k) = S(n-1,k-1) + kS(n-1,k) \]

Small values of \( S(n,k) \)

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Generating function of $S(n,k)$

Fix $k$, let $n$ vary: $F_k(x) = \sum_{n \geq k} S(n,k) x^n$

If $k \geq 1$, $\sum_{n \geq k} S(n,k) x^n = x \sum_{m \geq n-1} S(n-1,k-1) x^{n-1} + k x \sum_{n \geq k} S(n,k) x^{n-1}$

$= x \sum_{m \geq k-1} S(m,k-1) x^m + k x \sum_{m \geq k-1} S(m,k) x^m$

$= x F_{k-1}(x) + k x F_k(x)$

$\Rightarrow F_k(x) = x F_{k-1}(x) + k x F_k(x)$

$(1-kx) F_k(x) = x F_{k-1}(x)$

$F_k(x) = \frac{x}{1-kx} F_{k-1}(x)$

$F_0(x) = \sum_{n \geq 0} S(n,0) x^n = 1 \Rightarrow F_k(x) = \frac{x^k}{(1-kx)(1-(k-1)x) \cdots (1-x)}$ (Rational)

$\exists$ constants $a_{i,k}$ for $i=1, \ldots, k$ s.t.

$S(n,k) = \sum_{i=1}^{k} a_{i,k} i^n \text{ for } n \geq 1$

**Def.** Bell number $B(n) = \# \text{ partitions of set of size } n$

By definition, $B(n) = \sum_{k=0}^{n} S(n,k)$

$\sum_{n \geq 0} B(n) x^n = \sum_{k=0}^{\infty} F_k(x) = \sum_{k=0}^{\infty} \frac{x^k}{(1-x)(1-2x) \cdots (1-kx)}$
Thm. \( B(n+1) = \sum_{i=0}^{n} \binom{n}{i} B(i) \)

Pf. Separate partitions of \([n+1]\) based on how big block containing \(n+1\) is. Suppose size is \(j\).

How many such partitions?

Choose \(j-1\) numbers from \([n]\) to share w/ \(n+1\) \(\binom{n}{j-1}\)

Choose partition of remaining \(n-j+1\) numbers \(B(n-j+1)\)

\[1 \leq j \leq n+1 \] are the legal values:

\[ B(n+1) = \sum_{j=1}^{n+1} \binom{n}{j-1} B(n-j+1) \]

\[ = \sum_{i=0}^{n} \binom{n}{n-i} B(i) = \sum_{i=0}^{n} \binom{n}{i} B(i) \] \(\square\)