Linear Recurrence Relations

Setup. Sequence \((a_n)_{n \geq 0} = (a_0, a_1, a_2, \ldots)\) satisfies a (homogeneous) linear recurrence relation of order \(d\) if there exist constants \(c_1, \ldots, c_d\) with \(c_d \neq 0\) such that:

\[ a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_d a_{n-d} \quad \forall n \geq d. \]

Example (Fibonacci numbers) \(f_n = f_{n-1} + f_{n-2}\) \(\forall n \geq 2\)

\(f_0 = 0, f_1 = 1 \Rightarrow 0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots\)

Example \(d = 1\): \(a_n = c_1 a_{n-1} = c_1^2 a_{n-2} = \ldots = c_1^n a_0\)

Now consider \(d = 2\): \(a_n = c_1 a_{n-1} + c_2 a_{n-2}\) \(\forall n \geq 2\). (\(c_2 \neq 0\))

Def. The characteristic polynomial is \(t^2 - c_1 t - c_2\).

Let \(r_1, r_2\) be its roots \(\left(\frac{c_1 \pm \sqrt{c_1^2 + 4c_2}}{2}\right)\)

\(\Rightarrow t^2 - c_1 t - c_2 = (t - r_1)(t - r_2) \Rightarrow r_1 \neq 0, r_2 \neq 0.\)

Thus, if \(r_1 \neq r_2\), then \(\exists\) constants \(a_1, a_2\) such that:

\[ a_n = a_1 r_1^n + a_2 r_2^n \quad \forall n \geq 0 \]

To solve for \(a_1, a_2\), plug in \(n = 0, 1:\)

\(n = 0:\) \(a_0 = a_1 + a_2\)

\(n = 1:\) \(a_1 = a_1 r_1 + a_2 r_2\)
Example (Fibonacci) \( f_0 = 0, f_1 = 1, f_n = f_{n-1} + f_{n-2} \) \( n \geq 2 \)

char. poly \( t^2 - t - 1 \), \( r_1 = \frac{1 + \sqrt{5}}{2}, \ r_2 = \frac{1 - \sqrt{5}}{2} \)

\[ f_n = \alpha_1 \left( \frac{1 + \sqrt{5}}{2} \right)^n + \alpha_2 \left( \frac{1 - \sqrt{5}}{2} \right)^n \text{ for some } \alpha_1, \alpha_2 \]

\( n = 0 \): \( 0 = \alpha_1 + \alpha_2 \) \( \implies \) \( \alpha_1 = -\alpha_2 \)

\( n = 1 \): \( 1 = \alpha_1 r_1 + \alpha_2 r_2 \)
\( \implies \) \( \alpha_1 = \frac{1}{r_1 - r_2} = \frac{1}{\sqrt{5}} \)

\( \therefore f_n = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^n \quad \forall n \geq 0 \)

Example Periodic sequence \( x, y, x, y, x, y, \ldots \)

satisfies \( a_n = a_{n-2} \) \( \forall n \geq 2 \). char. poly is \( t^2 - 1 = (t-1)(t+1) \)

\( \implies \exists \alpha_1, \alpha_2 \text{ s.t. } a_n = \alpha_1 + \alpha_2(-1)^n \quad \forall n \geq 0 \)

\( n = 0 \): \( x = \alpha_1 + \alpha_2 \) \( \implies \alpha_1 = \frac{x+y}{2}, \ \alpha_2 = \frac{x-y}{2} \)

\( n = 1 \): \( y = \alpha_1 - \alpha_2 \)

\( \therefore a_n = \frac{x+y}{2} + (-1)^n \frac{x-y}{2} \)

Remark. Pick scalars \( c_1, c_2 \) s.t. \( t^2 - c_1 t - c_2 = (t - r_1)(t - r_2) \)

w/ \( r_1 + r_2 \) & \( c_2 \neq 0 \). Say a sequence \( (a_n) \) is a solution if \( a_n = c_1 a_{n-1} + c_2 a_{n-2} \) \( \forall n \geq 2 \). Let \( (a'_n) \) be another solution. Any linear combination \( (\lambda a_n + \delta a'_n) \) is also a solution.

\( \implies \) Set of solutions is a vector space (subspace of space of all sequences)
Then says: this subspace is spanned by \((r_1^n), (r_2^n)\)
\[\Rightarrow\text{Solution space is 2-dim'l}\]
But, \((r_1^n), (r_2^n)\) are linearly independent:
\[
(r_1^n) = (1, r_1, r_1^2, \ldots) \Rightarrow \text{solution space is 2-dim'l}
\]
\[
(r_2^n) = (1, r_2, r_2^2, \ldots)
\]

Why are \((r_1^n), (r_2^n)\) solutions?

Have to check that \(r_1^n = c_1 r_1^{n-1} + c_2 r_1^{n-2}\)
\[
r_1^n - c_1 r_1^{n-1} - c_2 r_1^{n-2} = r_1^{n-2}(r_1^2 - c_1 r_1 - c_2) = 0
\]

Every solution is determined by \(a_0, a_1\),
& these can be specified arbitrarily.