Math 188, Spring 2021
Homework 4
Due: April 30, 2021 11:59PM via Gradescope

Solutions must be clearly presented. Incoherent or unclear solutions will lose points.

Reminder: First part of project also due April 30. Last minute submissions run the risk of conflicting, in which case no credit is given.

http://www.math.ucsd.edu/~ssam/188/project.html

(1) Let \( n \) be a positive integer. Consider the equation
\[
x_1 + x_2 + \cdots + x_8 = 2n.
\]
For each of the following conditions, how many solutions are there? Give as simple of a formula as possible. (Each part is an independent problem, don’t combine the conditions.)
(a) The \( x_i \) are non-negative even integers.
(b) The \( x_i \) are positive odd integers.
(c) The \( x_i \) are non-negative integers and \( x_8 \leq 10 \).

(2) Let \( k,n \) be positive integers.
(a) Show that
\[
\sum_{(a_1,\ldots,a_n)} a_1 a_2 \cdots a_n = \binom{n+k-1}{k-n}
\]
where the sum is over all compositions of \( k \) into \( n \) parts. (Hint at end.)
(b) Show that
\[
\sum_{(a_1,\ldots,a_n)} 2^{a_2-1} 3^{a_3-1} \cdots n^{a_n-1} = S(k,n)
\]
where the sum is over all compositions of \( k \) into \( n \) parts.

(3) (a) Give a closed formula for the number of pairs of subsets \( S,T \) of \([n]\) such that \( S \not\subseteq T \) (i.e., \( S \subset T \) and \( S \neq T \)).
(b) Give a closed formula for the number of \( k \)-tuples of subsets \((S_1,\ldots,S_k)\) of \([n]\) such that \( \bigcap_{i=1}^{k} S_i = \emptyset \).

(4) (a) Let \( r \) be a fixed positive integer. Show that \( S(n+r,n) \) is a polynomial function of \( n \) of degree \( 2r \) for \( n \geq 0 \).
(b) Compute this polynomial for \( r = 2,3,4 \).

(5) Let \( F(n) \) be the number of set partitions of \([n]\) such that every block has size \( \geq 2 \). Prove that
\[
B(n) = F(n) + F(n+1),
\]
where \( B(n) \) is the \( n \)th Bell number.

1. Optional problems (don’t turn in)

(6) What is the total number of parts of all compositions of \( k \)?
[For example, when \( k = 2 \), the only compositions are \((2)\) and \((1,1)\) so there are a total of 3 parts.]

(7) Fix an integer \( k \geq 2 \). Call a composition \((a_1,\ldots,a_n)\) of \( k \) doubly even if the number of \( a_i \) which are even is also even (i.e., there could be no even \( a_i \), or 2 of them, or 4, etc.). Show that the number of doubly even compositions of \( k \) is \( 2^{k-2} \).
For example, if $k = 4$, then here are the 4 doubly even compositions of 4:

$$(2, 2), \quad (3, 1), \quad (1, 3), \quad (1, 1, 1, 1).$$

(8) Give a closed formula for the number of $k$-tuples of subsets $(S_1, \ldots, S_k)$ of $[n]$ such that $S_i \subseteq S_{i+1}$ for $i = 1, \ldots, k - 1$.

2. HINTS

(2)a: Consider the product

$$\left( \sum_{a_1 \geq 1} a_1 x^{a_1} \right) \cdots \left( \sum_{a_n \geq 1} a_n x^{a_n} \right).$$

(7): Given a composition $\alpha = (a_1, \ldots, a_n)$, define another composition $\Phi(\alpha)$ by

$$\Phi(\alpha) = \begin{cases} (1, a_1 - 1, a_2, a_3, \ldots, a_n) & \text{if } a_1 > 1 \\ (a_2 + 1, a_3, \ldots, a_n) & \text{if } a_1 = 1 \end{cases}.$$  

(in both cases, we didn’t do anything to $a_3, \ldots, a_n$). Show that $\Phi$ defines a bijection between the set of doubly even compositions of $k$ and the set of compositions of $k$ which are not doubly even.