Solutions must be clearly presented. Incoherent or unclear solutions will lose points.

(1) (a) Let \( r \) be a fixed nonnegative integer. Show that both \( S(n+r, n) \) and \( c(n+r, n) \) are polynomial functions of \( n \) of degree \( 2r \) for \( n \geq 0 \).

(b) Compute these polynomials for \( r = 2, 3 \).

(2) For \( n > 0 \), let \( a_n \) be the number of partitions of \( n \) such that every part appears at most twice, and let \( b_n \) be the number of partitions of \( n \) such that no part is divisible by 3. Set \( a_0 = b_0 = 1 \). Show that \( a_n = b_n \) for all \( n \).

(3) Let \( y \) be a variable. Prove the following generalization of Example 3.27

\[
\prod_{i \geq 0} \left(1 + x^{2i+1}y \right) = \sum_{r \geq 0} \frac{x^{r^2} y^r}{(1-x^2)(1-x^4) \cdots (1-x^{2r})}.
\]

Hint on next page.

(4) (a) Use the following \( q \)-analogue of Pascal’s identity (you don’t need to prove it)

\[
\begin{align*}
\binom{n}{k}_q &= q^n \binom{n-1}{k}_q + \binom{n-1}{k-1}_q \\
& \quad \text{for } n \geq k > 0
\end{align*}
\]

to show that if \( d \) is a non-negative integer, then

\[
\sum_{n \geq 0} \binom{n+d}{n}_q x^n = \prod_{i=0}^d (1-q^i x)^{-1} = \frac{1}{(1-x)(1qx) \cdots (1-q^d x)}.
\]

(b) Give a direct explanation (i.e., independent of the Schubert decomposition explanation from lecture) for why the coefficient of \( x^n \) of the right side is the sum \( \sum \lambda q^{\vert \lambda \vert} \) over all integer partitions \( \lambda \) whose Young diagram fits in the \( n \times d \) rectangle.

(5) Let \( V, W \) be \( F_q \)-vector spaces with \( \dim V = n \) and \( \dim W = m \).

(a) How many linear maps \( V \to W \) are there?

(b) Suppose \( n \geq m \). How many surjective linear maps \( V \to W \) are there?

(c) Pick \( k \leq \min(m, n) \). How many rank \( k \) linear maps \( V \to W \) are there?

See next page for some hints.
HINTS

3: If you expand this out, the coefficient of $x^n$ is a polynomial in $y$. If we set $y = 1$, this is just Example 3.27 and the coefficient of $x^n$ counts some special kind of partitions of $n$ on each side which are in bijection. By examining that proof more carefully, figure out what the coefficient of $y^k x^n$ means on each side and show that these descriptions match up under the bijection.

5a: Picking bases for $V$ and $W$, you can represent a linear map uniquely as a matrix.

5c: A rank $k$ linear map is essentially the same thing as a surjective linear map onto some $k$-dimensional subspace of $W$.

OPTIONAL PROBLEMS (DON’T TURN IN)

(6) This one is challenging (I can’t see a solution that is simple, but it doesn’t require anything outside of this class). Prove:

$$\sum_{n \geq 1} x^n(n-1)/2 = \prod_{n \geq 1} \frac{1 - x^{2n}}{1 - x^{2n-1}}.$$

(7) Pick integers satisfying $1 \leq k_1 < \cdots < k_r \leq n$. Let $X$ be the set of subspaces $W_1, \ldots, W_r$ of $F^n$ such that $\dim W_i = k_i$ for all $i$ and $W_i \subset W_{i+1}$ for $i < r$.

(a) Find a formula for $|X|$ using by generalizing Example 3.39, i.e., use a $q$-analogue of a multinomial coefficient.

(b) $|X|$ is also a polynomial in $q$; find an explicit description of this polynomial using a generalization of the Schubert decomposition of the Grassmannian.