(1) Let $F(x)$ be a formal power series with $F(0) = 0$.
   (a) Show that there exists a formal power series $G(x)$ with $G(0) = 0$ such that
       $F(G(x)) = x$ if and only if $[x^1]F(x) \neq 0$.
   (b) Assuming $[x^1]F(x) \neq 0$, show that $G(x)$ is unique and also satisfies $G(F(x)) = x$.
       You may use without proof that composition of formal power series is associative.

(2) Evaluate the following sums:
   (a) $\sum_{i=0}^{n} \binom{n}{i} \frac{1}{2^i}$
   (b) $\sum_{i=0}^{n} i^2 \binom{n}{i} 3^i$

(3) Let $a, b$ be non-negative integers.
   (a) By comparing coefficients in $(1 + x)^{a+b} = (1 + x)^a(1 + x)^b$, prove that for any
       non-negative integer $n$, we have
       $$\binom{a+b}{n} = \sum_{i=0}^{n} \binom{a}{i} \binom{b}{n-i}.$$  (Note: this is not as simple as it looks.)
   (b) Now prove this identity using a counting argument.
       [Hint: Consider choosing $n$ animals from $a$ dogs and $b$ cats...]

(4) How many ways can we arrange the letters of: MISSISSIPPI?

(5) Let $f(t) = \sum_{k=0}^{d} f_k t^k$ be a degree $d$ polynomial with rational coefficients. From
   lecture, we know that there exist unique rational numbers $g_0, \ldots, g_d$ such that
   $$\sum_{n \geq 0} f(n) x^n = \frac{g_0 + g_1 x + \cdots + g_d x^d}{(1 - x)^{d+1}}.$$  
   Now assume that $f(a)$ is an integer for $a = 0, \ldots, d$. (The $f_k$ don’t have to be
   integers for this to be true, for example $f(n) = n(n-1)/2$ has this property.)
   Prove that this implies that the $g_k$ are all integers and that $f(a)$ is an integer
   whenever $a$ is an integer.
   [Hint: first prove that
   $$f(t) = \sum_{k=0}^{d} g_k \binom{d+t-k}{d},$$
   as an identity of polynomials in $t$, and then consider the system of equations you get
   from $t = 0, \ldots, d$.]
(6) Let \( n \geq 2 \) be an integer.
(a) Prove that
\[
\sum_{i=0}^{n} i \binom{n}{i} (-1)^{i-1} = 0.
\]
(b) Compute
\[
\sum_{0 \leq i \leq n \atop i \text{ even}} i \binom{n}{i}.
\]

(7) (a) Let \( a, b \) be rational numbers. Show that for any formal power series \( A(x) \) with \( A(0) = 1 \), we have
\[
A(x)^a A(x)^b = A(x)^{a+b}.
\]
[Remember that we defined rational powers in a very specific way, so your proof needs to use this definition.]
(b) Deduce from (a) that
\[
\binom{a+b}{n} = \sum_{i=0}^{n} \binom{a}{i} \binom{b}{n-i}
\]
for all non-negative integers \( n \).

(8) Assume now that we deal with complex-coefficient formal power series. Define the following sets of formal power series:
\[
V = \{ F(x) \mid F(0) = 0 \},
\]
\[
W = \{ G(x) \mid G(0) = 1 \}.
\]
(a) Given \( F \in V \), show that \( E(F) = \sum_{n=0}^{\infty} F(x)^n \) is the unique formal power series \( G \in W \) such that \( DG = DF \cdot G \). This defines a function \( E : V \to W \).
[Convention: \( F(x)^0 = 1 \) even if \( F(x) = 0 \).]
(b) Given \( G \in W \), show that there is a unique formal power series \( F \in V \) such that \( DF(x) = DG(x)/G(x) \). We define the function \( L : W \to V \) by \( L(G) = F \).
[For the rest, it is unnecessary to use explicit formulas for \( L \) and \( E \) and in fact it may be easier to only use the uniqueness properties above.]
(c) Show that \( E \) and \( L \) are inverses of each other.
(d) Show that \( E(F_1 + F_2) = E(F_1)E(F_2) \) for all \( F_1, F_2 \in V \).
(e) Show that \( L(G_1G_2) = L(G_1) + L(G_2) \) for all \( G_1, G_2 \in W \).
(f) If \( m \) is a positive integer and \( G \in W \), show that \( E(\frac{G}{m^2}) \) is an \( m \)-th root of \( G \).
[This gives an alternative proof for the existence of \( m \)-th roots and in fact we can now define powers for any complex number \( m \): \( F^m = E(mL(F)) \).]
(g) Show that if \( \sum_{i \geq 0} F_i(x) \) converges to \( F(x) \), then \( \prod_{i \geq 0} E(F_i) \) converges to \( E(F) \).
(h) Show that if \( \prod_{i \geq 0} G_i(x) \) converges to \( G(x) \), then \( \sum_{i \geq 0} L(G_i) \) converges to \( L(G) \).