Solutions must be clearly presented. Incoherent or unclear solutions will lose points.

(1) Find a closed formula for the following recurrence relation:
\[ a_0 = 1, \quad a_1 = 1, \quad a_2 = 2, \]
\[ a_n = 5a_{n-1} - 8a_{n-2} + 4a_{n-3} \quad (n \geq 3). \]

(2) Find a closed formula for the following recurrence relation:
\[ a_0 = 1, \]
\[ a_n = 3a_{n-1} + 2^n \quad (n \geq 1). \]

(3) Let \( r_1, \ldots, r_d \) be distinct numbers. Prove that the sequences \( (r_{1}^n), \ldots, (r_{d}^n) \) are linearly independent by showing that the determinant of \( (r_{i,j}^{j-1})_{i,j=1,\ldots,d} \) is nonzero (interpret \( 0^0 = 1 \)).

(4) Let \( (a_n)_{n \geq 0} \) be a sequence satisfying a linear recurrence relation whose characteristic polynomial is \( (t^2 - 1)^d \).

(a) Show that there exist polynomials \( p(n) \) and \( q(n) \) of degree \( \leq d - 1 \) such that
\[ a_n = \begin{cases} p(n) & \text{if } n \text{ is even} \\ q(n) & \text{if } n \text{ is odd} \end{cases}. \]

(b) How does this generalize if the characteristic polynomial is \( (t^k - 1)^d \)?

(5) (a) Suppose that \( (a_n)_{n \geq 0} \) and \( (a'_n)_{n \geq 0} \) both satisfy the same linear recurrence relation of order \( d \) and that they agree in \( d \) consecutive places, i.e., there exists \( k \) such that \( a_k = a'_k, a_{k+1} = a'_{k+1}, \ldots, a_{k+d-1} = a'_{k+d-1} \). Show that these sequences are the same.

(b) Suppose that \( (a_n)_{n \geq 0} \) satisfies the linear recurrence relation of order \( d \)
\[ a_n = c_1a_{n-1} + \cdots + c_da_{n-d} \quad \text{for all } n \geq d. \]
Show that there is a unique sequence \( (b_n)_{n \in \mathbb{Z}} \) (indexed by all integers) such that \( b_n = a_n \) for \( n \geq 0 \) and such that
\[ b_n = c_1b_{n-1} + \cdots + c_db_{n-d} \quad \text{for all } n \in \mathbb{Z}. \]

Explain how to get a closed form formula for \( b_n \).

(c) Consider the Fibonacci sequence \( f_0 = 0, f_1 = 1, \) and \( f_{n+2} = f_{n+1} + f_n \). How does the negatively indexed Fibonacci sequence relate to the usual one?

1. Optional problems (don’t turn in)

(1) Let \( p \) be a prime number and let \( (a_n)_{n \geq 0} \) be a sequence such that \( a_n \in \mathbb{Z}/p \) and which satisfies a homogeneous linear recurrence relation. Show that the sequence is in fact periodic.

(2) Let \( r_1, \ldots, r_{d-1} \) be distinct numbers. Prove that the sequences \( \alpha_1 = (r_1^n), \ldots, \alpha_{d-1} = (r_{d-1}^n), \alpha_d = (nr_{d-1}^{n-1}) \) are linearly independent by showing that the determinant of \( (\alpha_{i,j-1})_{i,j=1,\ldots,d} \) is nonzero (interpret \( 0^0 = 1 \) and if \( r_{d-1} = 0 \), interpret \( \alpha_{d,0} = 0 \)).