\[ S(n,k) = S(n-1,k-1) + k S(n-1,k) \]

\[
\begin{array}{cccccc}
 n \backslash k & 1 & 2 & 3 & 4 & 5 \\
 1 & 1 & 0 & 0 & 0 & 0 \\
 2 & 1 & 1 & 0 & 0 & 0 \\
 3 & 1 & 3 & 1 & 0 & 0 \\
 4 & 1 & 7 & 6 & 1 & 0 \\
 5 & 1 & 15 & 25 & 10 & 1 \\
\end{array}
\]

\[ S(n,k) = 0 \text{ if } k > n \]

\[ B(n) = \# \text{ partitions of } [n], \quad n^{th} \text{ Bell number} \]

\[ = \sum_{k=0}^{n} S(n,k) \]

Therefore,

\[ B(n+1) = \sum_{i=0}^{n} \binom{n}{i} B(i) \]

**Proof.** Separate partitions of \([n+1]\) based on how big the block containing \(n+1\) is.

Let \(j\) be the size of block containing \(n+1\), \((1 \leq j \leq n+1)\). These partitions can be constructed as follows:

1. Pick \(j-1\) values from \([n]\) to share block w/ \(n+1\).
2. Partition remaining values \([n] \setminus \text{choices from (1)}\)

\[ \binom{n}{j-1} \text{ many ways to choose } j-1 \text{ values.} \]

\[ B(n-j+1) \text{ many ways to choose partition.} \]

\[ \Rightarrow \# \text{ partitions where block of } n+1 \text{ has size } j \text{ is} \]

\[ \binom{n}{j-1} B(n-j+1) \]
\[
\begin{align*}
\Rightarrow \quad B(n+1) &= \sum_{j=1}^{n+1} \binom{n}{j-1} B(n-j+1) = \sum_{k=0}^{n} \binom{n}{k} B(n-k) \\
&= \sum_{k=0}^{n} \binom{n-k}{k} B(n-k) = \sum_{i=0}^{n} \binom{n}{i} B(i). \quad \Box
\end{align*}
\]

**Integer partitions**

\(n = \) positive integer. An \textit{integer partition} of \(n\) is a sequence \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)\) of non-negative integers such that \(\lambda_1 + \ldots + \lambda_k = n\) and \(\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k \geq 0\).

- \(|\lambda| = n\) \textbf{size of } \lambda
- \(l(\lambda) = \# \{i : (\lambda_i > 0)\}\) \textbf{length of } \lambda

**Convention:** Partitions are considered the same if they differ only in 0's.

- \(p(n) = \# \text{ partitions of } n\)
- \(p_k(n) = \# \text{ partitions } \lambda \text{ of } n \text{ s.t. } l(\lambda) = k\)
- \(p_{\leq k}(n) = \# \text{ partitions } \lambda \text{ of } n \text{ s.t. } l(\lambda) \leq k\)

\(p(0) = 1\) by convention, let \(\emptyset\) denote unique partition of 0.

\begin{tabular}{ccc}
\text{EX.} & \(n\) & \text{partitions of } n \\
\hline
1 & \(\{1\}\) & 1 \\
2 & \(\{2\}, \{1,1\}\) & 2 \\
3 & \(\{3\}, \{2,1\}, \{1,1,1\}\) & 3 \\
4 & \(\{4\}, \{3,1\}, \{2,2\}, \{2,1,1\}, \{1,1,1,1\}\) & 5 \\
5 & \(\{5\}, \{4,1\}, \{3,2\}, \{3,1,1\}, \{2,2,1\}, \{2,1,1,1\}, \{1,1,1,1,1\}\) & 7
\end{tabular}
Visualize partitions using Young diagrams

\[ \lambda = (4, 2, 1), \quad Y(\lambda) = \begin{array}{c}
\Box \\
\Box \\
\Box \\
\end{array} \]

[Left-justified boxes.
\( \lambda_i \) boxes in \( i \)th row.]

\textbf{Transpose:} \( \lambda^T \) is partition w/ property that

\[ Y(\lambda^T) = \text{flip of } Y(\lambda) \text{ across diagonal.} \]

\[ \lambda^T = (3, 2, 1, 1) \]

\[ (\lambda^T)_i = \# \{ j \mid \lambda_j \geq i \}. \]

\[ (\lambda^T)^T = \lambda \]

\( \lambda \) is \underline{self-conjugate} if \( \lambda^T = \lambda \).

\textbf{Ex.} Self-conjugate partitions: (4, 3, 2, 1), (5, 1, 1, 1, 1), (4, 2, 1, 1)

```
\begin{array}{c}
\Box \\
\Box \\
\Box \\
\end{array} \quad \begin{array}{c}
\Box \\
\Box \\
\Box \\
\Box \\
\end{array} \quad \begin{array}{c}
\Box \\
\Box \\
\Box \\
\end{array}
```

\( \text{"staircase"} \quad \text{\( (\text{self-conjugate}) \) \"hook"} \)

\[ \#	ext{ partitions } \lambda \text{ of } n \text{ s.t. } \ell(\lambda) \leq k = \#	ext{ partitions } \mu \text{ of } n \text{ s.t. } \mu, \leq k \]

\[ P_{\leq k}(n) \]

\textbf{Pf.} Bijection between respective sets given by taking transpose.

\[ \begin{array}{c}
\{ \text{partition } \lambda \text{ of } n \mid \ell(\lambda) \leq k \} \\
\{ \text{partition } \mu \text{ of } n \mid \mu, \leq k \}
\end{array} \leftrightarrow \left\{ \begin{array}{c}
\lambda^T \\
g(\mu)
\end{array} \right\} \]

\[ f(\lambda) = \lambda^T, \quad g(\mu) = \mu^T. \]

If \( \ell(\lambda) \leq k \), then \( (\lambda^T)_i = \ell(\lambda) \leq k \)

If \( \mu, \leq k \), then \( \ell(\mu^T) = \mu, \leq k \).
Theorem: \( \text{# self-conjugate partitions} \leq \text{# partitions of } n \text{ using distinct odd parts} \)

**Example:**
- \( n = 10 \)
  - \( (5, 2, 1, 1, 1) \)
  - \( (4, 3, 2, 1) \)

**Diagram:**
- Bend into hooks

**Proof:**
- \( \{ \text{self-conjugate partitions} \} \leq \frac{1}{2} \{ \text{partitions of } n \text{ using distinct odd parts} \} \)

Let \( \lambda \) be self-conjugate.
- Remove \( \lambda_1 \) from \( \lambda \) and subtract 1 from remaining entries.
  - Removes \( 2\lambda_1 - 1 \) many boxes from \( Y(\lambda) \)
- We get smaller partition \( \mu = (\lambda_1 - 1, \lambda_2 - 1, \ldots) \)
- Remove \( \mu_1 \) from \( \mu \) and subtract 1 from remaining entries.
  - Removes \( 2\mu_1 - 1 \) many boxes from \( Y(\mu) \)
- Sequence \( f(\lambda) = (2\lambda_1 - 1, 2\mu_1 - 1, \ldots) \) is strictly decreasing, i.e., all distinct entries.

**Note:**
- \( 2\mu_1 - 1 = 2(\lambda_2 - 1) - 1 = 2\lambda_2 - 3 \leq 2\lambda_1 - 3 < 2\lambda_1 - 1 \).

**Constructing \( g \):** Start with \( \mu \), partition of \( n \) into odd distinct parts.
Each \( \mu_i \) is of the form \( 2x_i - 1 \) where \( x_i \geq 1 \).

Build self-conjugate hook of size \( \mu_i \) by taking

\[
\begin{pmatrix}
(x_i, 1, \ldots) \\
\overbrace{x_i - 1}^{x_i - 1}
\end{pmatrix}
\]

Take these hooks and nest them. Call this \( g(\mu_i) \),
(i.e. corner box of \( i^{th} \) hook goes in \( i^{th} \) row, \( i^{th} \) column)

Why partition?

\[
g(\mu_i) = (i - 1) + x_i - 1 = x_i + i - 1
\]

\[
g(\mu)_{i+1} = (i - 1) + x_i - 1 + x_{i+i} - 1 = x_{i+1} + i
\]

Know: \( 2x_i - 1 = \mu_i > \mu_{i+1} = 2x_{i+1} - 1 \)

\[
\implies x_i > x_{i+1}
\]

\[
x_i + i - 1 > x_{i+1} + i - 1 \implies x_{i+1} + i \geq x_{i+1} + i
\]

\[
g(\mu_i) \leq g(\mu)_{i+1}
\]