Strong induction

Strategy: ① Prove \( P(0) \)
② Use \( P(0), P(1), \ldots, P(n) \) to prove \( P(n+1) \)

**Example:** Want to prove every polynomial in \( X \) is a linear combination of powers of \( X-1 \):

\[ f(x), (x-1)^1, (x-1)^2, \ldots \]

let \( P(n) \) be "every polynomial of degree \( n \) is linear combination of powers of \( (x-1) \)"

① \( P(0) \): Every degree 0 polynomial is a constant \( c \),
\[ c = c \cdot 1 \checkmark \]

② Let \( f(x) \) be polynomial of degree \( n+1 \).

Let \( \alpha \) be leading coefficient of \( f(x) \).

\[ f(x) - \alpha(x-1)^{n+1} \text { has degree } \leq n \]

By induction \( \Rightarrow \) \( f(x) - \alpha(x-1)^{n+1} \) is linear combination of powers of \( (x-1) \).

\[ f(x) = \alpha x^{n+1} + \text{Smaller than degree } n \]

\[ -\alpha(x-1)^{n+1} = -\alpha x^{n+1} + \text{Smaller} \]

\[ f(x) - \alpha(x-1)^{n+1} = \text{Smaller than degree } n \]

\[ \Rightarrow \quad f(x) = \alpha x^{n+1} + \sum \alpha_i (x-1)^i \]
Def. \( S = \text{set}, \) permutation is an ordering of elements of \( S. \)

Ex. \( S = \{1, 2, 3\}, \) permutations are: \( \{1 \ 2 \ 3\}, \ \{1 \ 3 \ 2\}, \ \{2 \ 1 \ 3\}, \ \{2 \ 3 \ 1\}, \ \{3 \ 1 \ 2\}, \ \{3 \ 2 \ 1\}. \)

Def. \( 0! = 1, \) if \( n \) positive integers, \( n! = n \cdot (n-1)! \)

\[
\begin{align*}
n! & = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1 \\
6! & = 1 \cdot 1! = 2 \cdot 2! = 6 \cdot 3! = 24 \cdot 4! = 24 \cdot 5! = 120 \cdot 6! = 720
\end{align*}
\]

(\( \text{Assume \( n > 0 \).} \))

Thm. If \( |S| = n, \) then there are \( n! \) permutations of \( S. \)

pf. Induction on \( n. \)

\( n=1: \) one thing is ordered in unique way, \( 1! = 1 \) \( \checkmark \)

Assume known for \( n. \)

Let \( S \) be set of size \( n+1. \) To get a permutation,

Pick something to be first \( (n+1) \text{ choices} \)

Need to order remaining \( n \) elements \( (n!) \text{ ways by induction} \)

\[
(n+1) \cdot n! \text{ many ways to order everything} \quad \checkmark \)

Ex. 0. 2 red flowers, 1 black flower

\[
\begin{align*}
R_1 & R_2 B & R_1 & B & R_2 & B & R_1 & R_2 \\
R_2 & R_1 B & R_2 & B & R_1 & B & R_2 & R_1 & \left( \text{duplicates} \right)
\end{align*}
\]

Answer is: \( 3 \text{ ways to arrange.} \)

\[
\begin{align*}
\frac{3!}{2!} & \quad \text{or} \quad 3 \cdot 2 \cdot 1 \div 2 \cdot 1 = 3
\end{align*}
\]
2. 10 red flowers, 5 black flowers

Namely: \(15!\) permutations.

Redundant info: ordering of red flowers \(10!\)
ordering of black flowers \(5!\)

\[ \Rightarrow \frac{15!}{10! \cdot 5!} \]
many ways if flowers of
same color are treated same.

3. \(r\) red flowers, \(b\) black flowers:

\[ \frac{(r+b)!}{r! \cdot b!} \]

4. also: \(g\) green flowers:

\[ \frac{(r+b+g)!}{r! \cdot b! \cdot g!} \]

Total # of permutations:

\[ \binom{\# \text{ arrangements}}{\# \text{ ways to order flowers of same color}} \]

- \(n\) objects, each one has a "type" (color)
- \(k\) possible types, \(q_i\) objects of type \(i\).

Then, \(\#\) ways to arrange objects (treating same type as identical) is

\[ \frac{n!}{q_1! \cdot q_2! \cdots q_k!} \]

Def. Multinomial coefficient:
if \(q_1 + \cdots + q_k = n\), then

\[ \binom{n}{q_1, q_2, \ldots, q_k} = \frac{n!}{q_1! \cdot q_2! \cdots q_k!} \]
Special case \((k=2)\): \(\binom{n}{2}\) means \(\binom{n}{a_1,a_2}\)

**Words.**

**Def.** Let \(A\) be a set \((alphabet)\).

A word is a finite sequence of elements from \(A\).

Length is number of entries (could be 0).

**Ex.** \(A = \{a, b\}\) words of length \(\leq 2\):

\[
\emptyset, \quad a, \quad b, \\
\quad aa, \quad ab, \quad ba, \quad bb
\]

**Thm.** If \(|A|=n\), then there are \(n^k\) words of length \(k\).

Words of length \(k\) are \(k\)-tuples of elements of \(A\), i.e., elements of \(A^k = A \times A \times \ldots \times A\).

\([n] := \{1, \ldots, n\}\) for integer \(n \geq 0\).

**Ex.** We saw that \#subsets of \([n]\) = \(2^n\).

Let \(A = \{0, 1\}\)

Given a subset \(S \subseteq [n]\), define \(w_S\) as follows:

If \(i \in S\), then \(i\)'th entry of \(w_S\) is 1

If \(i \not\in S\), then \(i\)'th entry of \(w_S\) is 0
This gives \( f : \{ \text{subsets of } [n] \} \longrightarrow \{ \text{words in \{0,1\} of length } n \} \)
\[
f(S) = w_S
\]
Define inverse \( g \) as follows: given word \( w \), define
\[
g(w) \subseteq [n] \text{ by } g(w)_i = \{ \text{ indices } i \text{ s.t. } w_i = 1 \}
\]
f, g inverses of each other: so bijections
\[
\Rightarrow \quad \# \text{subsets of } [n] = \# \text{ words in } \{0,1\} \text{ of length } n = 2^n
\]

**EX.** How many choices of \( S, T \subseteq [n] \) s.t. \( S \subseteq T \)?

Let \( A = \{ \text{"in } S \text{ and } T", \text{"in } T \text{ but not } S", \text{"not in } T \text{ or } S" \} \)

\[
\begin{array}{ccc}
0 & 1 & 2 \\
\end{array}
\]

Given word in \( \{0,1\}^n \), get \( S, T \) as follows:

\[
\begin{align*}
S &= \{ \text{indices where } 0 \text{ appears} \} \\
T &= \{ \text{indices where } 0 \text{ or } 1 \text{ appears} \}
\end{align*}
\]

\[
\Rightarrow \quad f : \{ \text{words in } \{0,1\}^n \} \longrightarrow \{ \text{ \( S \subseteq T \subseteq [n] \) } \}
\]

\[
g(S \subseteq T) \text{ is word whose } i\text{th entry is } \begin{cases} 0 & \text{if } i \in S \\ 1 & \text{if } i \in T \setminus S \\ 2 & \text{if } i \notin T \end{cases}
\]

\Rightarrow \text{ bijection } \Rightarrow \text{ answer is } 3^n
\( n = 6 \)

\[ S = \{4,5\} \]

\[ T = \{1,4,5\} \]