A = alphabet of size k.
Count words of length n up to rotation (cyclic symmetry) the following are same:
Example: a_1, a_2, a_3, a_4, a_5, a_6
           a_2, a_3, a_4, a_5, a_6, a_1

Naïve guess: \( \frac{k^n}{n} \) (wrong)

Refinement: Note that 0101 has only 2 different rotations:
            1010
Given word \( w \), its period is least number of rotations of
\( w \) needed to get \( w \) back.
Ex. period of 0101 is 2.

Let \( w(d) = \# \) of words of period \( d \)

Observations:  period has to divide length of word.
              \( w(d) \) does not depend on \( n \)

Def. Equivalence class of word of length \( n \) up to cyclic symmetry
is necklace of length \( n \)

\[
\# \text{necklaces of } \frac{k^n}{d} \quad \frac{w(d)}{d} \quad \text{ex. } \frac{w(1)}{1} + \frac{w(2)}{2} + \frac{w(4)}{4}
\]

Second identity: \( \# \) words of length \( n \) up to cyclic symmetry
by using this identity, can try to solve for \( w(d) \).
Ex. Solve for \( \omega(4) \):

\[
\begin{align*}
\text{Start with} & \quad k^4 = \omega(1) + \omega(2) + \omega(4) \\
& \quad k^2 = \omega(1) + \omega(2) \\
& \quad k^4 - k^2 = \omega(4)
\end{align*}
\]

Solve for \( \omega(6) \):
\[
\begin{align*}
& \quad k^6 = \omega(1) + \omega(2) + \omega(3) + \omega(6) \\
& \quad k^3 = \omega(1) + \omega(3) \\
& \quad k^2 = \omega(1) + \omega(2) \\
& \quad k = \omega(1)
\end{align*}
\]
\[
\omega(6) = k^6 - k^3 - k^2 + k
\]

How to get general equation for \( \omega(n) \)?

Def. Define \( \mu(1) = 1 \). For \( n > 1 \)
\[
\mu(n) = \begin{cases} 
0 & \text{if } n \text{ is divisible by } p^2 \text{ for any prime } p \\
(-1)^r & \text{if } n \text{ is a product of } r \text{ different primes.}
\end{cases}
\]

Ex.
\[
\begin{align*}
\mu(1) &= 1 \\
\mu(2) &= -1 \\
\mu(3) &= -1 \\
\mu(4) &= 0 \quad \text{(since } 2^2 \text{ divides } 4) \\
\mu(5) &= -1 \\
\mu(6) &= (-1)^2 = 1 \\
\mu(7) &= -1 \\
\mu(8) &= 0 \quad \text{(since } 2^3 \text{ divides } 8) \\
\mu(9) &= 0 \\
\mu(10) &= (-1)^2 = 1 \\
\mu(11) &= -1 \\
\mu(12) &= 0 \\
12 &= 2^2 \cdot 3
\end{align*}
\]
Lemma. Let \( n \geq 1 \). Then \( \sum_{d \mid n} \mu(d) = 0 \)

Proof. Write prime factorization \( n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r} \) \((a_i \geq 1, p_i \text{ s different primes})\)

\[
\sum_{d \mid n} \mu(d) = \sum_{0 \leq e_1 \leq a_1} \mu(p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}) = \sum_{0 \leq e_1 \leq 1} \mu(p_1^{e_1}) \sum_{0 \leq e_2 \leq 1} \mu(p_2^{e_2}) \cdots \sum_{0 \leq e_r \leq 1} \mu(p_r^{e_r})
\]

\[
= \sum_{S \subseteq \{p_1, \ldots, p_r\}} \mu(S) = \sum_{S \subseteq \{p_1, \ldots, p_r\}} (-1)^{|S|} = \sum_{k=0}^{r} \binom{r}{k} (-1)^k = 0 \quad \text{for } r \geq 1.
\]

Theorem (Möbius Inversion). Let \( \alpha, \beta \) be complex-valued functions defined on positive integers.

1. If \( \alpha(d) = \sum_{e \mid d} \beta(e) \) for all \( d \),

Then \( \beta(d) = \sum_{e \mid d} \mu(d/e) \alpha(e) \) for all \( d \).

2. If \( \alpha(d) = \prod_{e \mid d} \beta(e) \) for all \( d \)

and \( \beta(e) \neq 0 \) for all \( e \), then \( \prod_{e \mid d} \mu(d/e) \alpha(e) \) for all \( d \).
Pf. Of C: \[ \sum \mu\left(\frac{d}{e}\right) \alpha(e) = \sum \mu\left(\frac{d}{e}\right) \sum \beta(f) \]
\[ = \sum \mu\left(\frac{d}{e}f\right) \beta(f) = \sum \mu(f) \sum \mu\left(\frac{d}{e}\right) \]
(f, e, r)
f \mid d 
eq \text{ s.t.}
\]

Have bijection:
\[ \left\{ e \mid \text{fle} \& \text{eld} \right\} \leftrightarrow \left\{ r \mid r \text{ divides } \frac{d}{f} \right\} \]
\[ e \mapsto \frac{d}{e} \]

Write prime factorization:
\[ d = p_1^{a_1} \cdots p_r^{a_r} \]
\[ f = p_1^{b_1} \cdots p_r^{b_r} \]

Since f \mid d, b_i \leq a_i \text{ for all } i.

If fle \& eld, then \[ e = p_1^{c_1} \cdots p_r^{c_r} \]
where \[ b_i \leq c_i \leq a_i \text{ for all } i. \]

Then:
\[ \frac{d}{e} = p_1^{a_1-c_1} p_2^{a_2-c_2} \cdots p_r^{a_r-c_r} \]
\[ \frac{d}{f} = p_1^{a_1-b_1} p_2^{a_2-b_2} \cdots p_r^{a_r-b_r} \]

\[ \left\{ e \mid \text{fle} \& \text{eld} \right\} \leftrightarrow \left\{ x \mid x \text{ divides } \frac{d}{f} \right\} \]
\[ \uparrow \]
\[ \left\{ (c_1, \ldots, c_r) \mid b_i \leq c_i \leq a_i \right\} \rightarrow \left\{ (e_1, \ldots, e_r) \mid 0 \leq e_1 \leq a_1-b_1 \right\} \]
\[ \left\{ b_1 \leq c_1 \leq a_1 \right\} \rightarrow \left\{ b_2 \leq c_2 \leq a_2 \right\} \rightarrow \left\{ b_r \leq c_r \leq a_r \right\} \]
\[ (c_1, \ldots, c_r) \rightarrow (c_1-b_1, \ldots, c_r-b_r) \]
\[
\sum_{\text{f,l.d.}} \sum_{\text{f,e,l.d.}} p(f) \mu\left(\frac{d}{e}\right) = \sum_{\text{f,l.d.}} p(f) \mu(x) \left[ \right] \quad \text{if } \frac{d}{e} > 1,
\]

\[\beta(d) \sum_{x \mid 1} \mu(11) = \beta(d) \]

For our problem: let \( p = 0 \).

\( \alpha(d) = k^d \)

Then \( \alpha(d) = \sum_{\text{e,l.d.}} \beta(e) \)

Höbius \( \beta(d) = \sum_{\text{e,l.d.}} \mu\left(\frac{d}{e}\right) \alpha(e) \)

Cor. For any positive integer \( d \),

\( \omega(d) = \sum_{\text{e,l.d.}} \mu\left(\frac{d}{e}\right) k^{e} \)

Ex. \# necklaces of length \( q \):

\[ \omega(1) = \sum_{\text{e,l.d.}} \mu\left(\frac{1}{e}\right) k^{e} = \mu(1) k = k \]

\[ \omega(2) = \mu\left(\frac{2}{1}\right) k^{1} + \mu\left(\frac{2}{2}\right) k^{2} = \mu(2) k + \mu(1) k^{2} = -k + k^{2} \]

\[ \omega(4) = \mu\left(\frac{4}{1}\right) k^{1} + \mu\left(\frac{4}{2}\right) k^{2} + \mu\left(\frac{4}{4}\right) k^{4} = \mu(4) k + \mu(2) k^{2} + \mu(1) k^{4} \]

\[ = \mu(4) k + \mu(2) k^{2} + \mu(1) k^{4} = k^{4} - k^{2} \]

\# necklaces of length \( q \) = \( \frac{\omega(1) + \omega(2) + \omega(4)}{2} = k + \frac{k^{2} - k}{2} + \frac{k^{4} - k^{2}}{4} \)
\[\omega^6(b) = \mu(b) \frac{b}{1} + \mu(b) \frac{b}{2} k^2 + \mu(b) \frac{b}{3} k^3 + \mu(b) \frac{b}{6} k^6 = k - k^2 - k^3 + k^6\]

**Notation:** \(\epsilon = \sqrt{-1}\)

**Euler’s identity:** \(e^{2\pi i} = 1\).

So solutions to \(x^n - 1 = 0\) are \(e^{2\pi i/n}, e^{2\pi i(1-1)/n}, \ldots, e^{2\pi i(n-1)/n}\), which are the \(n^{th}\) roots of unity.

If \(k, n\) have common factor \(r\), then \(e^{2\pi i k/n}\) is also \((\frac{n}{r})^{th}\) root of unity.

If \(k, n\) are relatively prime, then \(e^{2\pi i k/n}\) is called primitive \(n^{th}\) root of unity.

**Definition:** The \(n^{th}\) cyclotomic polynomial is

\[\Phi_n(x) = \prod_{k \text{ odd}, 0 \leq k \leq n-1 \land k, n \text{ relatively prime}} (x - e^{2\pi i k/n})\]

By definition, \(x^n - 1 = \prod_{d|n} \Phi_d(x)\)  

By Möbius inversion, \(\Phi_n(x) = \prod_{d|n} (x^{d-1})^\mu(n/d)\)
EX: \( n=6: \Phi_6(x) = \prod_{d|6} (x^{a_d-1})^{\mu(d/6)} = \frac{(x^6-1)(x-1)}{(x^2-1)(x^3-1)} = x^2 - x + 1. \)

\( n=8: \Phi_8(x) = \prod_{d|8} (x^{a_d-1})^{\mu(d/8)} = \frac{x^8-1}{x^4-1} = x^4 + 1 \)

**Crazy dice:**

Roll 2 6-sided dice, distribution for sum:

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<th>4</th>
<th>5</th>
<th>6</th>
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<td>8</td>
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**Question:** Can we relabel sides of dice so that the distribution of sum is as above?

Constraints must use positive integers.

One other solution:

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