labeled graphs: nodes 1,...,n w/ edges connecting them
\( \rightarrow \) \( 2 \) many
labeled forests: graphs w/ no cycles.
labeled trees: connected forests. \( t_n = \# \text{labeled trees} \)

Cayley: \( t_n = n^{n-2} \) for \( n \geq 1 \).

\[ t_n = \begin{cases} \text{for } n=3 & 1-2-3 \\ 2-1-3 \\ 2-3-1 \end{cases} \]

Ex: \( n=3 \)

\[ \begin{array}{c}
\circ \\
\circ \\
\circ \\
\end{array} \quad \begin{array}{c}
\circ \\
\circ \\
\circ \\
\end{array} \quad \begin{array}{c}
\circ \\
\circ \\
\circ \\
\end{array} \]

\( n=4 \)

\[ \begin{array}{c}
\circ \\
\circ \\
\circ \\
\circ \\
\end{array} \quad \begin{array}{c}
\circ \\
\circ \\
\circ \\
\circ \\
\end{array} \quad \begin{array}{c}
\circ \\
\circ \\
\circ \\
\circ \\
\end{array} \quad \begin{array}{c}
\circ \\
\circ \\
\circ \\
\circ \\
\end{array} \]

\( 4! / 2 \) = 12

Rooted labeled tree = pair \((T, i)\) \( T = \text{labeled tree on } n \text{ vertices} \)
\( 1 \leq i \leq n \)

\( \# \text{rooted labeled trees} = n t_n \)

Planted labeled forest = labeled forest + choice of root for each connected component

= disjoint union of labeled trees

(union of labels is \([n]\))

\( f_n := \# \text{planted labeled forests w/ } n \text{ vertices.} \quad (f_0 = 1) \)

\[ F(x) := \sum_{n=0}^{\infty} f_n \frac{x^n}{n!} \quad R(x) := \sum_{n=0}^{\infty} (n t_n) \frac{x^n}{n!} \]

Identity \( 1: \quad F(x) = e^{R(x)} \)
RF: To build planted labeled forest on \( n \) vertices:

1. Pick set partition \( X_1, \ldots, X_k \) of \( [n] \)
2. Put structure of rooted labeled tree on each \( X_i \)

This gives every planted labeled forest exactly once.

Exponential formula: \( F(x) = e^{R(x)} \)

Identity 2: \( R(x) = x F(x) \)

**Claim:** For \( n \geq 1 \), \( t_n = f_{n-1} \). Use bijection.

\[
\begin{align*}
\text{labeled trees on } [n] & \overset{f}{\longrightarrow} \text{planted labeled forests on } [n-1] \\
\text{on } [n] & \overset{g}{\longleftarrow}
\end{align*}
\]

Let \( T \) be labeled tree with vertices \( 1, \ldots, n \).

Delete vertex labeled \( n \) and all edges containing \( n \).

Result \( T' \) has no cycle, hence is a labeled forest on \( [n-1] \).

For each connected component of \( T' \), there is a unique vertex that was connected to \( n \) by an edge in \( T \). Let that be the root of the component.

Now I have planted labeled forest, call it \( f(T) \).
Let $U$ be planted labeled forest on $[n-1]$. Add vertex $n$. For each root in each component of $U$, add edge between $n$ and that root. Get labeled tree on $[n]$, call it $g(n)$.

$\Rightarrow f_g$ inverse, get bijection that proves $t_n = f_{n-1}$.

$$R(x) = \sum_{n\geq 1} (nt_n) \frac{x^n}{n!} = \sum_{n\geq 1} f_{n-1} \frac{x^n}{n!} = x \sum_{n\geq 1} \frac{x^{n-1}}{(n-1)!} = x F(x). \quad \square$$

$$R(x) = x F(x) = xe^{R(x)}$$

$$\Rightarrow \begin{cases} R(x) = x e^{R(x)} \end{cases}$$

Try to solve for cofficients of $R(x)$.

Let $r_n = \sum_{n\geq 0} \frac{nt_n}{n!}$ for $n > 0$, so $R(x) = \sum_{n\geq 0} r_n x^n$

$$r_0 = [x^0] R(x) = [x^0] (xe^{R(x)}) = [x^0] e^{R(x)}$$

Notes: Constant term of $R(x)$ is 0

so $[x^0] e^{R(x)} = [x^0] (1 + \frac{R(x)}{1!} + \frac{R(x)^2}{2!} + \cdots) = 1$
\[ r_2 = [x^2] R(x) = [x^2] (x e^{R(x)}) = [x^1] e^{R(x)} \]
\[ = [x^1] \left( 1 + R(x) + \frac{R(x)^2}{2!} + \frac{12R(x)^3}{3!} + \ldots \right) = r_1 = 1 \]
\[ r_3 = [x^3] R(x) = [x^3] (x e^{R(x)}) = [x^2] e^{R(x)} \]
\[ = [x^2] \left( 1 + R(x) + \frac{R(x)^2}{2!} + \frac{R(x)^3}{3!} + \ldots \right) \]
\[ = r_2 + \frac{1}{2!} (r_0 r_2 + r_1^2 + r_2 r_3) = 1 + \frac{1}{2} (0 + 1 + 3) = \frac{3}{2} \cdot \]

**Lagrange Inversion Formula**: Let \( G(x) \) be FPS with nonzero constant term. Then there is a unique FPS \( A(x) \) s.t.
\[ A(x) = x G(A(x)) \).
Furthermore, \([x^0] A(x) = 0 \) and for \( n > 0 \),
\[ [x^n] A(x) = \frac{1}{n} [x^{n-1}] (G(x)^n) \).

**Conclusion of Cayley's Formula**:
Use Lagrange inversion w/ \( G(x) = e^x \).
For \( n > 0 \),
\[ [x^n] R(x) = \frac{1}{n} [x^{n-1}] e^x = \frac{1}{n} [x^{n-1}] \sum_{k=0}^{n} \frac{x^k}{k!} \cdot \]
Set \( k = n - 1 \):
\[ = \frac{1}{n} \frac{n^{n-1}}{(n-1)!} \]
By definition, \[ \binom{n^m}{n} = \frac{n^m}{m!} \]

\[ \Rightarrow \quad \frac{n^m}{m!} \quad \Rightarrow \quad t_n = n^{n-2} \]

(Other method: Prüfer encoding)

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Catalan numbers (again)

Recall: \[ C(x) = \sum_{n \geq 0} C_n x^n \]

We showed: \[ C(x) = 1 + x C(x)^2 \]

Define \[ A(x) = C(x) - 1 \]

\[ \Rightarrow \quad A(x) \neq \mathbf{1} + x (A(x) - 1)^2 \]

\[ \Rightarrow \quad A(x) = x (A(x) + 1)^2 \]

Can apply Lagrange inversion w/ \( G(x) = (x+1)^2 \)

For \( n > 0 \),

\[ [x^n] A(x) = \frac{1}{n} \left[ x^{n-1} \right] (x+1)^{2n} = \frac{1}{n} \left[ x^{n-1} \right] \sum_{k=0}^{2n} \binom{2n}{k} x^k \]

\[ = \frac{1}{n} \left( \binom{2n}{n-1} \right) \]

Note: \[ \frac{1}{n} \left( \binom{2n}{n-1} \right) = \frac{(2n)!}{n!(n-1)!(n+1)!} = \frac{1}{n+1} \left( \frac{(2n)!}{n! \cdot n!} \right) = \frac{1}{n+1} \binom{2n}{n} . \]

Also, for \( n > 0 \), \[ [x^n] A(x) = [x^n] C(x) . \]
Generalized Catalan:

Recall: \( C_n = \# \text{ rooted binary trees w/ } n+1 \text{ leaves} \) 
\[ = \# \text{ rooted binary trees w/ } n \text{ internal vertices} \]

Let \( k \geq 2 \) integer.

Consider \( \# \text{ rooted } k\text{-ary trees w/ } n \text{ internal vertices} \)

Let's call this \( \alpha_n \)

\[ \Rightarrow \alpha_n = \sum_{(i_1, i_2, \ldots, i_k)} \alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_k} (x) \text{ for } n \geq 0 \]

\[ i_1 + \cdots + i_k = n-1 \]

\[ \Rightarrow \text{ Define } B(x) = \sum_{n \geq 0} \alpha_n x^n. \]

(*) Translates to 
\[ B(x) = 1 + x B(x)^k \]

Define \( A(x) = B(x) - 1 \):

\[ A(x) = x (A(x) + 1)^k \]

Use Lagrange w/ \( G(x) = (x+1)^k. \)

\[ [x^n] A(x) = \frac{1}{n} \left[ x^{n-1} \right] (x+1)^{nk} = \frac{1}{n} \left[ x^{n-1} \right] \sum_{i=0}^{nk} \binom{nk}{i} x^i \]

\[ = \frac{1}{n} \binom{nk}{n-1} \]
\[ B(x) = \sum_{n \geq 0} b_n x^n. \]

What is coeff of \( x^n \) in \( B(x)^k \)?

Claim: \( \sum_{i+j=n} b_i b_j \)

When \( k=2 \), \( \sum_{i=0}^{n} b_i b_{n-i} = \sum_{(i,j)} b_i b_j \)

Prove by induction.

For \( k > 2 \), \( B(x)^k = B(x)^{k-1} B(x) \)

\[ (x^n) B(x)^k = (x^n) (B(x)^{k-1} B(x)) \]

\[ = \sum_{j=0}^{n} [x^j] B(x)^{k-1} b_{n-j} \]

\[ = \sum_{j=0}^{n} \sum_{(i_1, \ldots, i_{k-1})} b_{i_1} \cdots b_{i_{k-1}} b_{n-j} \]

Every weak composition of \( n \) into \( k \) parts is of the form \( (i_1, \ldots, i_{k-1}, n-j) \) where \( i_1 + \cdots + i_{k-1} = j \) for some \( 0 \leq j \leq n \).