What is this case about?

3 sample problems

1. Linear recurrence relations:
   Fibonacci sequence: 0, 1, 2, 3, 5, 8, 13, ...
   Can I find formula (non-recursive)?

2. Counting # ways to write n pairs of balanced parentheses
   \( n=2: \ ( () \ ) \ ( () ) \)
   \( n=3: \ ( () ( ) \ ) \ ( () () ) \ ( ( () ) \ ( ( ) ) \ ) \ ( ( ) ) \ ( ( ) ) \ ( ( ) ( ) ) \)
   general \( n \)?

3. How many permutations of \( n \) things have no fixed points?

   **Bijection**
   Given sets \( X, Y \), functions \( f: X \rightarrow Y \)
   \( g: Y \rightarrow X \)

   we say inverses of each other if
   \( f \circ g = id_Y \) & \( g \circ f = id_X \)

   i.e., For all \( y \in Y \), \( f(g(y)) = y \), For all \( x \in X \), \( g(f(x)) = x \)

   If so, \( f \) and \( g \) are bijections.

   **Prop.** If a bijection exists between \( X, Y \) then \( |X| = |Y| \).

   If \( f \) is bijection, then it is
   - one-to-one (injective), i.e., if \( f(x) = f(x') \), then \( x = x' \)
   - onto (surjective), i.e., for all \( y \in Y \), there is some \( x \)
     so that \( f(x) = y \)

   \( f \) is bijection \( \iff \) injective & surjective.
For injective \( f \Rightarrow |X| \leq |Y| \)

For surjective \( f \Rightarrow |X| \geq |Y| \)

**Sum principle:** X, Y sets w/ no overlap.

Then \( |X \cup Y| = |X| + |Y| \).

**Product principle:** \( X \times Y = \{ (x, y) \mid x \in X, y \in Y \} \)

\( |X \times Y| = |X| \cdot |Y| \)

12 -fold way: Assigning \( k \) balls to \( n \) boxes.

1. Subject to some condition:
   Think of assignment as function \( f: \{ \text{balls} \} \rightarrow \{ \text{boxes} \} \)
   Conditions on \( f \):
   1. \( f \) is injective
   2. \( f \) is surjective
   3. no condition

   Conditions on \text{balls}:
   1. all identical
   2. considered different

   Conditions on \text{boxes}:
   1. all identical - indistinguishable
   2. considered different - distinguishable
<table>
<thead>
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<th>balls/boxes</th>
<th>f arbitrary</th>
<th>f injective</th>
<th>f surjective</th>
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**Weak Induction:** Prove sequence of statements \( P(0), P(1), P(2), \ldots \)

**Strategy:**
1. Prove \( P(0) \) is true. **Base case**
2. Use \( P(n) \) to prove \( P(n+1) \) is true. **Inductive step**

**Example:** \( P(n) \) is statement \( \sum_{i=0}^{n} i = \frac{n(n+1)}{2} \)

1. \( P(0): \left( \sum_{i=0}^{0} i = \frac{0(0+1)}{2} = 0 \right) \) \( \checkmark \)
2. \( \sum_{i=0}^{n+1} i = \sum_{i=0}^{n} i + (n+1) = \frac{n(n+1)}{2} + (n+1) = (n+1) \left( \frac{n}{2} + 1 \right) = (n+1) \left( \frac{n+2}{2} \right) \)
   \( P(n) \Rightarrow P(n+1) \) \( \checkmark \)

**Subsets.** A set, \( T \) is a subset of every element of \( T \) belongs to \( S \).

**Note:** \( T \) could be empty, and also \( T = S \) \( \checkmark \).
Thm. If \(|S| = n\), there are \(2^n\) subsets of \(S\).

Proof:
Base case: \(n = 0\), \(S = \emptyset\), only subset is \(T = \emptyset\)
\[2^0 = 1 \quad \checkmark\]

Inductive step: Suppose \(|S| = n+1 > 0\).

Pick some element \(x \in S\). Consider 2 kinds of subsets:

1: \(T\) contains \(x\)
2: \(T\) does not contain \(x\)

How many of type II?

Define \(f: \) subsets of \(S\) \(\rightarrow\) subsets of \(S \setminus \{x\}\)
\[f(T) = T \setminus \{x\}\]
get bijection \(g\)
\[g(T) = T\]

\(\Rightarrow 2^n\) subsets of type II.

Ex. \(S = \{a, b, c\}\), \(x = b\)

Subsets of \(S\) not containing \(b\):
- \(\emptyset\)
- \(\{a\}\)
- \(\{b, c\}\)
- \(\{c\}\)

Subsets of \(S \setminus \{x\}\):
- \(\emptyset\)
- \(\{a\}\)
- \(\{c\}\)

How many of type I?

\[f(T) = T \setminus \{x\}\]
\[g(U) = U \cup \{x\}\]

bijection \(\Rightarrow 2^n\) type I subsets.

\[\#\text{subsets} = \#\text{type I} + \#\text{type II} = 2^n + 2^n = 2(2^n) - 2^{n+1}\]
\[ g(f(T)) = g(T \setminus x) = (T \setminus x) \cup \{x\} \]

\[ f(g(U)) = f(U \cup \{x\}) = U \]

Ex.

\[ \sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \]

\[ \sum_{i=0}^{n} i^3 = ?? \quad \text{deg 4 polynomial in } n \]

\[ \sum_{i=0}^{n} i^d = \text{deg } d+1 \text{ polynomial in } n. \]

\[ S = \{a, b, c\}, \quad x = b \]

\[ \{\} \]

\[ \{a\} \]

\[ \{c\} \]

\[ \{a, c\} \]

\[ \{a, b, c\} \]

\[ \{a, c\} \]