Math 184, Fall 2023
Homework 5
Due: Wednesday, Nov. 15 by 11:59PM via Gradescope (late homework will not be accepted)

Explanations should be given for your solutions when appropriate. Use complete sentences.
I'll put some hints and answers to check your work on the next page.

(1) Consider the alphabet consisting of the 3 symbols `(` `)` `0`. Call a word **balanced** if, when deleting the `0`'s, the result is a balanced set of parentheses (empty word allowed). Let a_n be the number of balanced words of length n.
For instance, $a_4 = 9$ and the set of balanced words of length 4 are:

`0000`, `()00`, `(0)0`, `(00)`, `0()0`, `0(0)`, `00()`, `()()`, `(())`.

(a) Prove that (a_n) satisfies the following recursive formula:

$$a_0 = a_1 = 1, \qquad a_n = a_{n-1} + \sum_{i=0}^{n-2} a_i a_{n-2-i} \qquad (\text{for } n \geq 2).$$

(b) Use the technique in the proof of Corollary 6.3.3 and the text following it to find a simple formula for $A(x) = \sum_{n \geq 0} a_n x^n$, i.e., find a quadratic polynomial that $A(x)$ is a root of and then use the quadratic formula to solve for $A(x)$.
You do not need to solve for a closed formula for a_n.

(2) (a) Let b_n be the number of paths from $(0, 0)$ to $(2n, 0)$ using the steps $\overrightarrow{(1, 1)}$ and $\overrightarrow{(1, -1)}$ which never go strictly below the x-axis (touching x-axis is ok). Prove that $b_n = C_n$, the nth Catalan number and draw the 5 examples when $n = 3$.
(b) Consider the sequence a_n from Question 1. Give an interpretation of a_n as counting certain paths from $(0, 0)$ to $(n, 0)$ (you tell me what steps are allowed and what conditions to impose) and explain why your interpretation is correct.

(3) (a) We have n distinguishable cars. How many ways can we paint each one either red, blue, or green such that #(red cars) is even and #(blue cars) is odd.
(b) Continuing with the situation in (a), we add the colors white and yellow, but with the rule that #(white cars) + #(yellow cars) must be an even number (0 allowed). How many ways can this be done?

(4) Fix a positive integer k.
(a) Let a_n be the number of set partitions of $[n]$ into k blocks such that every block has even size (0 is not allowed since blocks are required to be nonempty!). Give a simple expression for the EGF $A(x) = \sum_{n \geq 0} a_n \frac{x^n}{n!}$.
(b) Let b_n be the number of set partitions of $[n]$ into k blocks such that all blocks have size ≥ 2. Give a simple expression for the EGF $B(x) = \sum_{n \geq 0} b_n \frac{x^n}{n!}$.

(5) Let α be the structure defined so that $\alpha(S)$ has size 1, and its single element is the identity function on S. Let β be the structure defined so that $\beta(S)$ is the set of all bijections $f : S \to S$ such that $f(x) \neq x$ for all $x \in S$.
Let $\gamma = \alpha \cdot \beta$ and explain why $|\gamma([n])| = n!$. Use this to prove that

$$E_\beta(x) = \frac{e^{-x}}{1 - x}.$$

Checking your work

(1b): If you plug in $x = 2$ into your final expression (this has no mathematical significance related to the original problem, but it does allow to check your answer), you should get $-\frac{1+\sqrt{-15}}{8}$.

(3): if you take $n = 6$, you should get (a) 182 and (b) 1862.

Hints

Q1–Q4 use ideas that are similar to examples in class, so there isn't much more to add.

For Q5: if we label the elements of S as $1, \dots, |S|$, then we can think of the identity function as a permutation such that all cycles have length 1. Similarly, we can think of the elements of $\beta(S)$ also as permutations. What condition is $f(x) \neq x$ for all x imposing on the cycle lengths of f?