Math 184, Winter 2022
Homework 4
Due: Friday, Feb. 11 by 11:59PM via Gradescope (late homework will not be accepted)

Explanations should be given for your solutions. Use complete sentences.

1. If \( \sum_{n \geq 0} a_n x^n = \frac{2 + 3x^2 - 2x^3}{(1 - 5x)^5} \), find a closed formula for \( a_n \).
2. Define a sequence by
   \[
   a_0 = 1, \quad a_1 = 3, \quad a_n = 8a_{n-1} - 16a_{n-2} + 3^n \text{ for } n \geq 2.
   \]
   (a) Express \( A(x) = \sum_{n \geq 0} a_n x^n \) as a rational function in \( x \).
   (b) Find a closed formula for \( a_n \).
3. Let \( S(n, k) \) be the Stirling number of the second kind. For each \( k \geq 1 \), define the ordinary generating function
   \[
   S_k(x) = \sum_{n \geq 0} S(n, k)x^n.
   \]
   (a) For \( k \geq 2 \), translate the identity from lecture
   \[
   S(n, k) = S(n - 1, k - 1) + k \cdot S(n - 1, k)
   \]
   into an identity involving \( S_k(x) \) and \( S_{k-1}(x) \).
   (b) Use the identity you found in (a) and induction on \( k \) to show that for all \( k \geq 1 \):
   \[
   S_k(x) = \frac{x^k}{(1 - x)(1 - 2x) \cdots (1 - kx)}.
   \]
4. You want to build a stack of blocks that is \( n \) feet high. You have 3 different kinds (unlimited of each): green blocks are 1 foot high, while red and blue blocks are 2 feet high. Blocks of the same color are considered indistinguishable. Let \( a_n \) be the number of ways to stack these blocks.
   Find a linear recurrence relation and initial conditions satisfied by \( a_n \).
5. You are designing a race that takes place over \( n \) blocks in a city. It will consist of 3 portions: running, followed by biking, and ending with another running portion. The end of a portion should match up with the end of a block. The first running portion needs to designate 3 blocks to have a first aid tent, and the biking portion needs to designate 4 blocks to have a first aid tent. The second running portion doesn’t need anything, but must have positive length. Use generating functions to find a formula for the number of ways to design a race under these conditions.
6. Let \( n \) be a positive integer and let \( a_n \) be the number of different ways to pay \( n \) dollars using only 1, 2, 5, 10, 20 dollar bills in which at most three 20 dollar bills are used. Express \( A(x) = \sum_{n \geq 0} a_n x^n \) as a rational function.