CHAPTER



Changing the Form of a Function

Most limit problems that you'll meet will include a fractional expression in which the numerator and denominator have a common factor that needs to be canceled out. Moral: you need to know how to factor to be a player in this game. It isn't always obvious how you should factor, but in your factoring toolbox you've got four basic methods that should be tried in this order:

1) Common factors
2) Special formulas
3) Grouping
4) The Factor Theorem

Recall that factoring an expression means writing it as a product.

5.1 Common Factors

The easiest tool to use is to take out all factors that are common to all the terms of the expression.

EXAMPLE 1 Factor $3x^2y^3 + 15xy^4 - 21x^3y^2$.

Solution Notice that all three terms share a factor of 3, as well as x and y^2. So, taking out this common factor of $3xy^2$, we get

$$3x^2y^3 + 15xy^4 - 21x^3y^2 = 3xy^2(xy + 5y^2 - 7x^2). \quad \blacksquare$$

Remarks This is the easiest method of factoring and should always be done before going on to the rest.

5.1 Exercises

Factor the following expressions:

1) $2xy + 4x$
2) $6wz + 2wzt$
3) $xy + 4x + 2xw$
4) $6x^2y + 3xy + 9xy^2$
5) $10x^8y^6 + 25x^2y^4 + 20x^3y^{10}$
6) $24x^2yz + 2xy^2z^2 + 4xyz^3$

5.2 Special Formulas

1) $x^2 - y^2 = (x + y) \cdot (x - y)$

2) $x^3 + y^3 = (x + y) \cdot (x^2 - xy + y^2)$

3) $x^3 - y^3 = (x - y) \cdot (x^2 + xy + y^2)$

4) $x^2 + (a + b)x + ab = (x + a) \cdot (x + b)$

5) $acx^2 + (bc + ad)x + bd = (ax + b) \cdot (cx + d)$

6) $x^2 + 2xy + y^2 = (x + y)^2$ and $x^2 - 2xy + y^2 = (x - y)^2$

The formulas of 6 are special cases of the two previous formulas. They are called **perfect squares.**

Remark You should know formulas 1, 2, 3, and 4 "in your sleep." The others, especially 5, are not so critical. All of them can be checked by multiplying out the right-hand sides.

Formula 1 is called the **difference of two squares** and comes up a lot. Consider the following examples.

EXAMPLE 1 a) $z^2 - 9 = (z + 3) \cdot (z - 3)$

Here we are using 1 with $x = z$ and $y = 3$.

b) $x^4 - y^2 = (x^2)^2 - y^2 = (x^2 + y) \cdot (x^2 - y)$

Here x is replaced by x^2.

c) $(x - y)^2 - 4y^2 = (x - y + 2y) \cdot (x - y - 2y)$
$= (x + y) \cdot (x - 3y)$ ■

Formula 2 is called the **sum of two cubes**, while formula 3 is the **difference of two cubes**. Notice how similar they are. One way to remember the signs in a formula is as follows: whatever sign is between the two cubes matches the sign between x and y in the first factor of the product, while the sign in front of the xy in the second factor is opposite to that. (Notice the $-xy$ in the middle of formula 2 and the xy in the middle of formula 3. It sometimes surprises people.)

EXAMPLE 2 a) $a^3 + 8b^3 = a^3 + (2b)^3 = (a + 2b) \cdot (a^2 - 2ab + 4b^2)$

b) $27x^3 + 64y^3z^6 = (3x)^3 + (4yz^2)^3$
$= (3x + 4yz^2) \cdot (9x^2 - 12xyz^2 + 16y^2z^4)$

Here we are using formula 2 with x replaced by $3x$ and y replaced by $4yz^2$.

c) $a^3 + 2b^3$ is the sum of the cubes of a and $\sqrt[3]{2}b$ (which equals $2^{1/3}b$). So $a^3 + 2b^3 = (a + 2^{1/3}b) \cdot (a^2 - 2^{1/3}ab + 2^{2/3}b^2)$. ■

The usage of formula 3 goes pretty much the same way as that of formula 2.

EXAMPLE 3 $x^3 - 64y^6 = (x - 4y^2) \cdot (x^2 + 4xy^2 + 16y^4)$
Again, watch the sign of that middle term in the last factor! ■

Formula 4 is used by noticing that the middle term is the **sum** of a and b, while the last term is their **product.**

EXAMPLE 4 Factor $x^2 + 5x + 6$.

Solution We are looking for two numbers a and b so that their product is 6 and their sum is 5. Well, 6 factors only as 2 times 3, 6 times 1, -2 times -3, or -6 times -1. Of all these possibilities the only pair to add up to 5 is 2 and 3.

So: $$x^2 + 5x + 6 = (x + 2) \cdot (x + 3).$$ ■

EXAMPLE 5 Factor $y^2 - 10y + 21$.

Solution We need two numbers whose product is 21, so we think of 3 and 7, -3 and -7, 21 and 1, and -21 and -1. However, we notice that their sum must be -10, so we use -3 and -7:

$$y^2 - 10y + 21 = (y - 3) \cdot (y - 7).$$ ■

EXAMPLE 6 Factor $s^2 + 2s - 8$.

Solution Now we need two numbers whose product is -8 and whose sum is 2. The number -8 is the product of -8 and 1, -4 and 2, 4 and -2, or 8 and -1. The correct pair that adds up to 2 is 4 and -2, hence

$$s^2 + 2s - 8 = (s + 4) \cdot (s - 2).$$ ■

EXAMPLE 7 Factor $a^4 - a^2 - 6$.

Solution This looks different, but if we let $x = a^2$, we get $x^2 - x - 6$, and then this method of factoring applies. Now, -6 is the product of -6 and 1, -3 and 2, -2 and 3, and -1 and 6. Since their sum has to be -1, the only possible choice is -3 and 2. So,

$$x^2 - x - 6 = (x - 3) \cdot (x + 2)$$

and hence

$$a^4 - a^2 - 6 = (a^2 - 3) \cdot (a^2 + 2).$$

This can be factored further to give

$$a^4 - a^2 - 6 = (a - \sqrt{3})(a + \sqrt{3})(a^2 + 2).$$ ■

Remark In the last example the factorization to $(a^2 - 3) \cdot (a^2 + 2)$ is as far as you can go if you factor "over the integers," meaning that all the coefficients must be integers. If you factor "over the reals," you must go the extra step and get $(a - \sqrt{3})(a + \sqrt{3})(a^2 + 2)$. In some other contexts, you can even factor $a^2 + 2$ "over the complex numbers" into $(a + \sqrt{2}i)(a - \sqrt{2}i)$. In most situations in calculus, you'll need to factor only over the reals.

EXAMPLE 8 Factor $y^2 + 10y - 24$.

Solution Notice how -24 is the product of many pairs, so this method of factoring can get to be tedious—in fact it can be a real pain! Since their sum is 10, the pair that works is 12 and -2, so

$$y^2 + 10y - 24 = (y - 2) \cdot (y + 12). \quad \blacksquare$$

Ray of Hope: If this method is too tedious, or too tough, or if the factors are not whole numbers (which happens often), you can use a different method based on the quadratic formula and the Factor Theorem (F.T.). For example, $x^2 - 2x - 4$ can't be factored easily without the Factor Theorem, which we will see in Section 5.4.

Formula 5 is usually really tough to use because too many possibilities need to be checked, unless you're lucky and hit the right one quickly. Your best bet in this case is to factor out the coefficient of the first term and work with the remaining expression. And there's always the Factor Theorem.

5.2 Exercises

Factor the following expressions. (Here, to factor means to factor as far as possible over the reals; see Example 7.)

1) $4y^2 - 9z^2$
2) $16x^2 - 36y^2$
3) $16x^4 - y^6$
4) $16x^4 - y^4$
5) $8s^3 + 27t^3$
6) $27s^3 + 64t^3$
7) $2x^3 + 64y^3$
8) $8s^3 - 27t^3$
9) $64z^3 - 9t^3$
10) $64y^6 - z^6$
11) $x^2 + 2x + 1$
12) $x^2 + 6x + 8$
13) $x^2 - 2x - 24$
14) $a^4 - 2a^2 - 24$
15) $s^6 - 7s^3 - 8$
16) $s^6 - 26s^3 - 27$
17) $3x^2 + x - 2$
18) $2x^2 + x - 3$
19) $-2x^2 + 8x - 8$
20) $x^2y^2z + 2xy^2z + y^2z$



5.3 Grouping

This method sometimes, but not always, does the job. Here's how it works.

EXAMPLE 1 Factor $10xy + 15y + 4x + 6$.

Solution As written, there are no common factors, but notice that the first and second terms have a common factor of $5y$, while the third and fourth terms have a common factor of 2, giving

$$(10xy + 15y) + (4x + 6) = 5y(2x + 3) + 2(2x + 3).$$

Now these two terms have a common factor of $2x + 3$, which we factor out to get

$$10xy + 15y + 4x + 6 = (2x + 3) \cdot (5y + 2).$$

Notice that the expression is in factored form. ■

EXAMPLE 2 Factor $6ax + 3ay - 4bx - 2by + 10x + 5y$.

Solution You can group in more than one way. Let's try

$$\begin{aligned}(6ax + 3ay) + (-4bx - 2by) + (10x + 5y)\\ &= 3a(2x + y) + (-2b)(2x + y) + 5(2x + y)\\ &= (2x + y) \cdot (3a - 2b + 5).\end{aligned}$$ ■

Remarks

a) In the above example, we could have grouped differently. Noticing that some terms have an x-factor and some a y-factor, we could have done this:

$$\begin{aligned}6ax + 3ay - 4bx - 2by + 10x + 5y\\ &= (6ax - 4bx + 10x) + (3ay - 2by + 5y)\\ &= 2x(3a - 2b + 5) + y(3a - 2b + 5)\\ &= (3a - 2b + 5)(2x + y).\end{aligned}$$

The resulting factorization is the same as before. Always, if grouping is going to work, the resulting factorization will be the same regardless of how you group.

b) Not all expressions can be factored by grouping; in fact, some expressions cannot be factored **by any method**. If an expression has a prime number of terms, it definitely can't be factored by **grouping**. (Can you explain why?)

c) The resulting factors may need further factoring afterward.

5.3 Exercises

Factor the following completely:

1) $3ax + 2ay + 3bx + 2by$

2) $2ax + 3ay + 2bx + 3by$

3) $x^4 - x^3y + x - y$

4) $x^5 - x^4y + 2x - 2y$

5) $x^{10} + x^6y^2 + x^4y^3 + y^5$

6) $x^5 - x^3y^2 - x^2y^3 + y^5$

7) $6x^3y - 4xy^3 + 12yx^2 - 8y^3$

8) $3x^3y + 2xy^3 - 6yx^2 - 4y^3$

9) $3x^2 + 5xy + 7x + 3xy + 5y^2 + 7y$

5.4 The Factor Theorem and Long Division

Here it comes, the long-awaited entrance of the ultimate in theoretical sensations. Yes folks, the one, the only, the celebrated Factor Theorem (F.T.). Here is the pearl of wisdom. (Let it rip, Frank!)

Factor Theorem

Let $P(x)$ be a polynomial. Let a be any real number. Then $x - a$ is a factor of $P(x)$ if and only if $P(a) = 0$.

Remark So what's the big deal with this factor theorem? Here's what. Suppose you want to factor a polynomial $P(x)$, and the other methods don't look promising. But suppose you can find some number a such that $P(a) = 0$. (You can do this by trying easy numbers like $a = \pm 1, \pm 2$, etc. to see if you are lucky and find such an a. If you have a graphing calculator, you can see where the graph of the function crosses or touches the x-axis. By zooming in on one such point, you may obtain a number a where $P(a) = 0$.) **If you can find** an a with $P(a) = 0$, then you are sure, because of the F.T., that $x - a$ is a factor of $P(x)$. So $P(x) = (x - a) \cdot (\text{something})$. That "something" can be found by long division, and so you've factored $P(x)$!

EXAMPLE 1 Factor $P(x) = x^3 - 2x^2 - 5x + 6$.

Solution This one looks tough. The earlier methods of factoring would give you nothing. Zip. Zero. Nada. Goose eggs. But notice that if you put $x = 1$, you get $1^3 - 2 \cdot 1^2 - 5 \cdot 1 + 6 = 0$. Aha! So the F.T. says that $x - 1$ is a factor. So

$$P(x) = (x - 1) \cdot (\text{something}).$$

You're half done. To get the other factor, use long division:

$$x - 1\overline{)x^3 - 2x^2 - 5x + 6}$$

(Notice that both expressions are in descending powers!)

Divide the first term into the first term (x into x^3 gives x^2). Put the x^2 on top, multiply it by the **entire divisor** $x - 1$, and subtract it from the dividend $x^3 - 2x^2 - 5x + 6$:

$$\begin{array}{r} x^2 \\ x - 1\overline{)x^3 - 2x^2 - 5x + 6} \\ \underline{x^3 - x^2} \\ -x^2 - 5x + 6 \end{array}$$

The easiest way to subtract $x^3 - x^2$ is to **mentally** change the sign of $x^3 - x^2$, getting $-x^3 + x^2$, and **add.** Continue in the same way: x into $-x^2$ gives $-x$, etc., to get:

$$\begin{array}{r} x^2 - x \\ x - 1\overline{)x^3 - 2x^2 - 5x + 6} \\ \underline{x^3 - x^2} \\ -x^2 - 5x + 6 \\ \underline{-x^2 + x} \\ -6x + 6 \end{array}$$

One more time: x into $-6x$ gives -6. So:

$$\begin{array}{r} x^2 - x - 6 \\ x - 1\overline{)x^3 - 2x^2 - 5x + 6} \\ \underline{x^3 - x^2} \\ -x^2 - 5x + 6 \\ \underline{-x^2 + x} \\ -6x + 6 \\ \underline{-6x + 6} \\ 0. \end{array}$$

We knew that the remainder would be 0 because the F.T. told us so.

Hence $P(x) = (x - 1) \cdot (x^2 - x - 6)$.

So $P(x)$ is factored, but not factored completely as yet because $(x^2 - x - 6)$ can be factored further to give

$$P(x) = (x - 1) \cdot (x - 3) \cdot (x + 2).$$

Success! ■

EXAMPLE 2 Factor $2x^2 + 3x - 2$.

Solution You could try special formula 5, but it would take too long. Remembering the F.T., all you need is a number a that makes $2a^2 + 3a - 2 = 0$. In ordinary English: you need a solution to $2x^2 + 3x - 2 = 0$. But that's easy, if you recall the quadratic formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

where in this context, $a = 2, b = 3$, and $c = -2$. Using this formula, you get $x = \dfrac{-3 \pm \sqrt{9 + 16}}{4} = \dfrac{-3 \pm 5}{4} = \dfrac{1}{2}$ or -2. So the F.T. gives us both factors $\left(x - \frac{1}{2}\right)$ and $(x + 2)$, and so

$$2x^2 + 3x - 2 = \left(x - \frac{1}{2}\right) \cdot (x + 2) \cdot (\text{something}).$$

Since the coefficient of the x^2 term on the left is 2, that something **must be 2**, for equality. So

$$2x^2 + 3x - 2 = \left(x - \frac{1}{2}\right)(x + 2)(2) = (2x - 1) \cdot (x + 2). \quad ■$$

EXAMPLE 3 Can $x^2 + x + 1$ be factored?

Solution The F.T. tells us to check for solutions of the equation $x^2 + x + 1 = 0$. The quadratic formula gives

$$x = \frac{-1 \pm \sqrt{1 - 4}}{2},$$

which is not a real number, because negative numbers don't have square roots among the real numbers. (See Section 1.5 for further explanation.) So there is no real number a to make $a^2 + a + 1$ equal to 0. Hence the F.T. says that $x^2 + x + 1$ **can't be factored**. (Such quadratics are called **irreducible**.) ■

To sum up: You can always use the F.T. to factor quadratics by using the quadratic formula, but for other polynomials $P(x)$, you'll need to be lucky to find a number a that makes $P(a) = 0$. Try $a = \pm 1, \pm 2$, etc.

In real-life problems, scientists and engineers use calculators or computers to find, or approximate, values for a where $P(a) = 0$.

5.4 Exercises

Factor the following expressions, if possible:

1) $x^3 - 3x + 2$
2) $x^3 - 7x + 6$
3) $2x^3 + x + 3$
4) $x^3 - x + 6$
5) $x^2 - 3x - 2$
6) $2x^2 - 3x + 4$
7) $2x^2 - 5x + 4$
8) $24x^2 - 48x - 72$
9) $x^3 - 3x - 2$
10) $x^3 + 2x^2 - 13x + 10$

5.5 Rationalizing Numerators or Denominators Using Conjugates

Consider the expression $\dfrac{x - \sqrt{2}}{5}$. Notice that the numerator has two terms, one of which is a square root. For various reasons, you may not want a square root in the numerator. You can always get rid of it by multiplying and dividing by its conjugate $x + \sqrt{2}$. (You find the **conjugate** by changing the sign between the two terms.) So you will get

$$\frac{x - \sqrt{2}}{5} = \frac{x - \sqrt{2}}{5} \cdot \frac{x + \sqrt{2}}{x + \sqrt{2}} = \frac{(x - \sqrt{2}) \cdot (x + \sqrt{2})}{5 \cdot (x + \sqrt{2})}$$

$$= \frac{x^2 - 2}{5 \cdot (x + \sqrt{2})}.$$

Remarks

a) The top is now root-free.

b) The little devil's conjugate has popped up in the denominator instead. Depending on the problem you're solving, that may not cause any difficulties.

c) It all boils down to this: you can **exchange** a square root in the top for a square root in the bottom, or vice versa, whichever is better in a particular case.

EXAMPLE 1 Rationalize the denominator of $\dfrac{x^2-3}{x+\sqrt{3}}$.

Solution This means get rid of the root in the bottom. So multiply both top and bottom by the conjugate $x-\sqrt{3}$, giving

$$\frac{x^2-3}{x+\sqrt{3}} = \frac{x^2-3}{x+\sqrt{3}} \cdot \frac{x-\sqrt{3}}{x-\sqrt{3}} = \frac{(x^2-3)(x-\sqrt{3})}{(x+\sqrt{3})(x-\sqrt{3})}$$

$$= \frac{(x^2-3)(x-\sqrt{3})}{(x^2-3)} = (x-\sqrt{3}). \quad \blacksquare$$

Remark When you're working with a fractional expression, and either the top or bottom has two terms, one (or both) of which is a square root, it is often useful to rationalize it. This comes up in limits but also in many other cases. If you have such an expression, it **should always pop into your mind that one option is to rationalize**, which may help.

EXAMPLE 2 Rationalize $\dfrac{x^4-25}{x-\sqrt{5}}$.

Solution Multiply both top and bottom by the conjugate $x+\sqrt{5}$, giving

$$\frac{x^4-25}{x-\sqrt{5}} = \frac{x^4-25}{x-\sqrt{5}} \cdot \frac{x+\sqrt{5}}{x+\sqrt{5}} = \frac{(x^4-25)(x+\sqrt{5})}{(x^2-5)},$$

which can be factored to:

$$= \frac{(x^2-5)(x^2+5)(x+\sqrt{5})}{(x^2-5)} = (x^2+5)(x+\sqrt{5}),$$

which is probably easier to deal with. ■

Remark You may have noticed that you can get the same result by factoring the top completely and then canceling:

$$\frac{x^4-25}{x-\sqrt{5}} = \frac{(x^2+5)(x^2-5)}{x-\sqrt{5}} = \frac{(x^2+5)(x+\sqrt{5})(x-\sqrt{5})}{(x-\sqrt{5})} = (x^2+5)(x+\sqrt{5}).$$

That's true, but you've still got to know how to rationalize (see 6.1 Exercises).

EXAMPLE 3 Rationalize $\dfrac{\sqrt{x+h}-\sqrt{x}}{h}$.

Solution Multiply both top and bottom by the conjugate $\sqrt{x+h}+\sqrt{x}$, giving:

$$\frac{\sqrt{x+h}-\sqrt{x}}{h}=\frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h(\sqrt{x+h}+\sqrt{x})},$$

which when simplified gives

$$\frac{\sqrt{x+h}-\sqrt{x}}{h}=\frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})}=\frac{1}{(\sqrt{x+h}+\sqrt{x})}.$$

The following example presents part of the calculation needed to find the derivative of $f(x)=\dfrac{1}{\sqrt{x}}$.

EXAMPLE 4 Write $\dfrac{1}{\sqrt{x+h}}-\dfrac{1}{\sqrt{x}}$ as one fraction, and rationalize the resulting numerator.

Solution First, use a common denominator to get

$$\frac{\sqrt{x}-\sqrt{x+h}}{\sqrt{x+h}\sqrt{x}}.$$

Now multiply both top and bottom by the conjugate $\sqrt{x}+\sqrt{x+h}$.

$$\begin{aligned}\frac{\sqrt{x}-\sqrt{x+h}}{\sqrt{x+h}\sqrt{x}}&=\frac{\sqrt{x}-\sqrt{x+h}}{\sqrt{x+h}\sqrt{x}}\cdot\frac{\sqrt{x}+\sqrt{x+h}}{\sqrt{x}+\sqrt{x+h}}\\&=\frac{x-(x+h)}{\sqrt{x+h}\sqrt{x}(\sqrt{x}+\sqrt{x+h})}\\&=\frac{-h}{\sqrt{x+h}\sqrt{x}(\sqrt{x}+\sqrt{x+h})}\ \blacksquare\end{aligned}$$

5.5 Exercises

In Exercises 1–7, rationalize the top or bottom, and simplify.

1) $\dfrac{7}{\sqrt{2}-1}$

2) $\dfrac{x+\sqrt{2}}{2}$

3) $\dfrac{3}{x-\sqrt{7}}$

4) $\dfrac{x+1}{x+\sqrt{11}}$

5) $\dfrac{x^2-3}{x-\sqrt{3}}$

6) $\dfrac{x^4-36}{x+\sqrt{6}}$

7) $\dfrac{x^8 - 9}{x^2 + \sqrt{3}}$

8) Let $f(x) = \dfrac{1}{\sqrt{2x}}$. Calculate $\dfrac{f(x+h) - f(x)}{h}$ and simplify as in Example 4.

5.6 Extracting Factors from Radicals

Radicals can be difficult when computing things, so it usually pays to make them as simple as possible. Extracting factors from under the radical sign is one way of simplifying.

EXAMPLE 1 Simplify $\sqrt[3]{250x^4y^3}$.

Solution Since $(ab)^n = a^n \cdot b^n$ for all n (as long as both sides are defined!), we have $(ab)^{1/3} = a^{1/3} \cdot b^{1/3}$ and $\sqrt[3]{a \cdot b} = \sqrt[3]{a} \cdot \sqrt[3]{b}$. In this example, you can write $250x^4y^3 = (125x^3y^3)(2x)$. (We chose the first factor to be a **perfect cube**.) So $\sqrt[3]{250x^4y^3} = \sqrt[3]{125x^3y^3}\sqrt[3]{2x} = 5xy\sqrt[3]{2x}$. ■

EXAMPLE 2 Simplify the radical by extracting all that you can from $\sqrt{25x^8y}$.

Solution
$$\begin{aligned}\sqrt{25x^8y} &= \sqrt{(25x^8)y} \\ &= \sqrt{25x^8}\sqrt{y} \\ &= 5x^4\sqrt{y}\end{aligned}$$ ■

Remarks

a) Notice that in Example 1 as you "pulled" the factor $125x^3y^3$ out of the radical, it "became" its cube root $5xy$. Similarly, as you "pulled" $25x^8$ out of the radical in Example 2, it "changed into" its square root $5x^4$.

b) When working with square roots and other even-powered roots, you must remember that $\sqrt{a^2} = |a|$, not just a. The next example will illustrate.

EXAMPLE 3 Simplify $\sqrt{36xy^2z^3}$.

Solution $\sqrt{36xy^2z^3} = \sqrt{(36y^2z^2)(xz)}$, where the first factor is a perfect square.

So
$$\begin{aligned}\sqrt{36xy^2z^3} &= \sqrt{(36y^2z^2)}\sqrt{(xz)} \\ &= (\sqrt{36}\sqrt{y^2}\sqrt{z^2})\sqrt{xz} \\ &= 6|y||z|\sqrt{xz}.\end{aligned}$$ ■

EXAMPLE 3 Rationalize $\dfrac{\sqrt{x+h}-\sqrt{x}}{h}$.

Solution Multiply both top and bottom by the conjugate $\sqrt{x+h}+\sqrt{x}$, giving:

$$\frac{\sqrt{x+h}-\sqrt{x}}{h}=\frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h(\sqrt{x+h}+\sqrt{x})},$$

which when simplified gives

$$\frac{\sqrt{x+h}-\sqrt{x}}{h}=\frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})}=\frac{1}{(\sqrt{x+h}+\sqrt{x})}.$$

The following example presents part of the calculation needed to find the derivative of $f(x)=\dfrac{1}{\sqrt{x}}$.

EXAMPLE 4 Write $\dfrac{1}{\sqrt{x+h}}-\dfrac{1}{\sqrt{x}}$ as one fraction, and rationalize the resulting numerator.

Solution First, use a common denominator to get

$$\frac{\sqrt{x}-\sqrt{x+h}}{\sqrt{x+h}\sqrt{x}}.$$

Now multiply both top and bottom by the conjugate $\sqrt{x}+\sqrt{x+h}$.

$$\begin{aligned}\frac{\sqrt{x}-\sqrt{x+h}}{\sqrt{x+h}\sqrt{x}}&=\frac{\sqrt{x}-\sqrt{x+h}}{\sqrt{x+h}\sqrt{x}}\cdot\frac{\sqrt{x}+\sqrt{x+h}}{\sqrt{x}+\sqrt{x+h}}\\&=\frac{x-(x+h)}{\sqrt{x+h}\sqrt{x}(\sqrt{x}+\sqrt{x+h})}\\&=\frac{-h}{\sqrt{x+h}\sqrt{x}(\sqrt{x}+\sqrt{x+h})}\end{aligned}$$ ■

5.5 Exercises

In Exercises 1–7, rationalize the top or bottom, and simplify.

1) $\dfrac{7}{\sqrt{2}-1}$

2) $\dfrac{x+\sqrt{2}}{2}$

3) $\dfrac{3}{x-\sqrt{7}}$

4) $\dfrac{x+1}{x+\sqrt{11}}$

5) $\dfrac{x^2-3}{x-\sqrt{3}}$

6) $\dfrac{x^4-36}{x+\sqrt{6}}$

7) $\dfrac{x^8 - 9}{x^2 + \sqrt{3}}$

8) Let $f(x) = \dfrac{1}{\sqrt{2x}}$. Calculate $\dfrac{f(x+h) - f(x)}{h}$ and simplify as in Example 4.

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Solution Since $(ab)^n = a^n \cdot b^n$ for all n (as long as both sides are defined!), we have $(ab)^{1/3} = a^{1/3} \cdot b^{1/3}$ and $\sqrt[3]{a \cdot b} = \sqrt[3]{a} \cdot \sqrt[3]{b}$. In this example, you can write $250x^4y^3 = (125x^3y^3)(2x)$. (We chose the first factor to be a **perfect cube**.) So $\sqrt[3]{250x^4y^3} = \sqrt[3]{125x^3y^3}\sqrt[3]{2x} = 5xy\sqrt[3]{2x}$. ■

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Solution
$$\begin{aligned}\sqrt{25x^8y} &= \sqrt{(25x^8)y} \\ &= \sqrt{25x^8}\sqrt{y} \\ &= 5x^4\sqrt{y}\end{aligned}$$ ■

Remarks

a) Notice that in Example 1 as you "pulled" the factor $125x^3y^3$ out of the radical, it "became" its cube root $5xy$. Similarly, as you "pulled" $25x^8$ out of the radical in Example 2, it "changed into" its square root $5x^4$.

b) When working with square roots and other even-powered roots, you must remember that $\sqrt{a^2} = |a|$, not just a. The next example will illustrate.

EXAMPLE 3 Simplify $\sqrt{36xy^2z^3}$.

Solution $\sqrt{36xy^2z^3} = \sqrt{(36y^2z^2)(xz)}$, where the first factor is a perfect square.

So
$$\begin{aligned}\sqrt{36xy^2z^3} &= \sqrt{(36y^2z^2)}\sqrt{(xz)} \\ &= (\sqrt{36}\sqrt{y^2}\sqrt{z^2})\sqrt{xz} \\ &= 6|y||z|\sqrt{xz}.\end{aligned}$$ ■

EXAMPLE 4 Simplify $\sqrt[4]{16x^8y^3}$.

Solution
$$\begin{aligned}\sqrt[4]{16x^8y^3} &= \sqrt[4]{16x^8}\sqrt[4]{y^3} \\ &= \sqrt[4]{16}\sqrt[4]{x^8}\sqrt[4]{y^3} \\ &= 2|x^2|\sqrt[4]{y^3}\end{aligned}$$

Here, the stuff in the absolute value sign, x^2, is always nonnegative, and so $|x^2| = x^2$. So the given expression equals

$$2x^2\sqrt[4]{y^3}. \quad \blacksquare$$

EXAMPLE 5 Simplify $\sqrt[5]{32x^{10}y^2}$.

Solution $\sqrt[5]{32x^{10}}\sqrt[5]{y^2} = 2x^2\sqrt[5]{y^2}$ ■

EXAMPLE 6 Simplify $\sqrt{x^2y^6 + 3x^5y^4}$.

Solution The stuff in the radical is not factored yet, so you must do that before you can extract any factors.

$$\begin{aligned}\sqrt{x^2y^6 + 3x^5y^4} &= \sqrt{x^2y^4(y^2 + 3x^3)} \\ &= \sqrt{x^2y^4}\sqrt{y^2 + 3x^3} \\ &= |x||y^2|\sqrt{y^2 + 3x^3} \\ &= |x|y^2\sqrt{y^2 + 3x^3} \quad \blacksquare\end{aligned}$$

5.6 Exercises

Extract as much as you can from the following roots:

1) $\sqrt{16x^2}$

2) $\sqrt{4x^2 + 8x^4}$

3) $\sqrt[3]{54x^4}$

4) $\sqrt{3x^{12}y}$

5) $\sqrt{5x^4 + 3x^8}$

6) $\sqrt[3]{27x^6y}$

7) $\sqrt{8\pi^2x^3y^4}$

8) $\sqrt[3]{x^7y^4 + x^6y^5}$

9) $\sqrt[4]{x^5y^4 + x^6y^{10}}$