CHAPTER

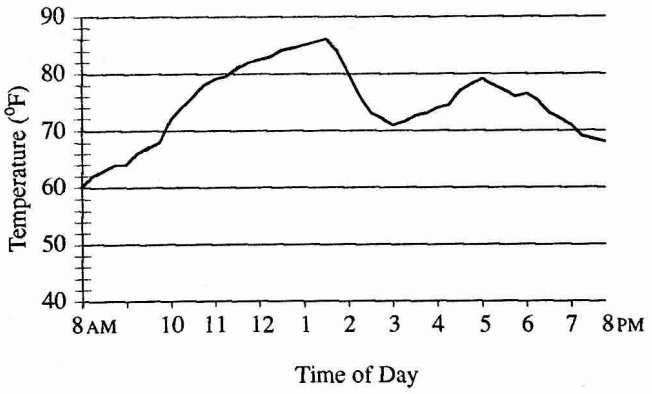
4

Functions and Their Graphs

4.1 Introduction

In calculus, we are concerned with quantities whose values are not constant but rather depend on the value of some other quantity, often time. For example, the temperature displayed in Times Square in New York City changes quite a bit, but it has a particular value at any given point in time. We say the temperature is a **function** of time.

EXAMPLE 1 Here is a record of the temperature in Times Square between 8 AM and 8 PM on a certain day:



a) Approximately what was the temperature at noon? b) What was the highest temperature, and when did it occur? c) What was the lowest, and when did it occur? d) Give a rough estimate of the average temperature over that 12-hour interval. e) What possible explanations could there be for the dip in the temperature in midafternoon?

Solution a) At noon, the temperature was roughly 83°F.
b) The highest temperature was about 86°F, at 1:30 PM.
c) The lowest temperature was about 60°F, at 8 AM.
d) The average was obviously less than 80°F and greater than 70°F, maybe around 75°F.

e) Here are some possibilities: a passing shower, shade from a tall building, King Kong blocking the sunlight. ■

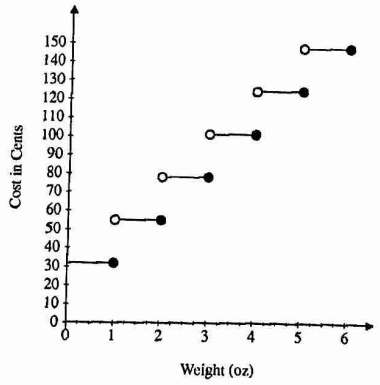
The graph shown in the previous example is sometimes called a **line graph,** even though it is obviously not a line. Quantities that vary as a function of another quantity can always be represented by such a graph. Another way that a function can be represented is in the form of a table, and in using such a representation it is important to know how to translate the data into a graph.

EXAMPLE 2 We're off to the post office to mail a letter. We check the cost of mailing a letter first-class and find this table:

Weight Not Over (oz)	Cost
1	$ 0.33
2	0.55
3	0.77
4	0.99
5	1.21
6	1.43

Here, the cost of mailing a first-class letter is a function of how much the letter weighs. Take the tabular representation of the function and graph cost as a function of weight.

Solution



Notice that the open circles indicate missing points, while the solid circles indicate included points. For example, if the weight is exactly 1 ounce, the cost is 33 cents, not 55 cents. Also notice that this is a line graph, although a strange one. The function is said to be **piecewise defined.** ■

So we see that functions can be defined by tables and by graphs. They can also be defined by words and by symbolic expressions. This fact is sometimes referred to as the *Rule of Four.* It is important that you can easily change functional representations from any one form to another. Consider the next example.

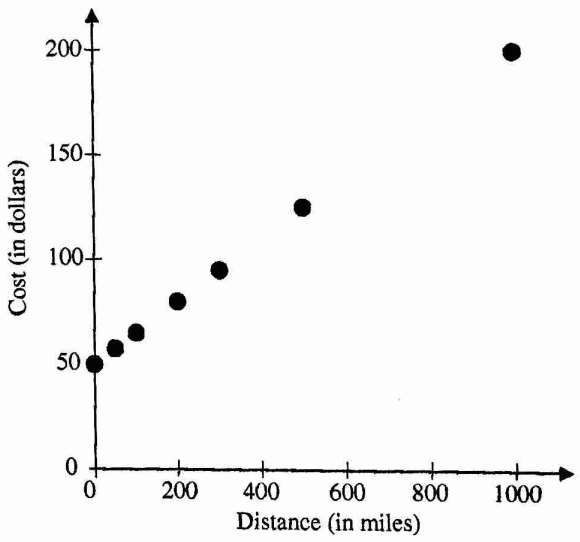
EXAMPLE 3 Hurts Car Rental charges \$50 for a weekend, plus 15 cents per mile driven. Let C be the rental cost, in dollars, and let d be the distance driven, in miles. a) Express C symbolically as a function of d. b) Graph C as a function of d.

Solution a) $C = 50 + (\text{number of miles driven})(0.15)$
$C = 50 + (0.15)(d)$

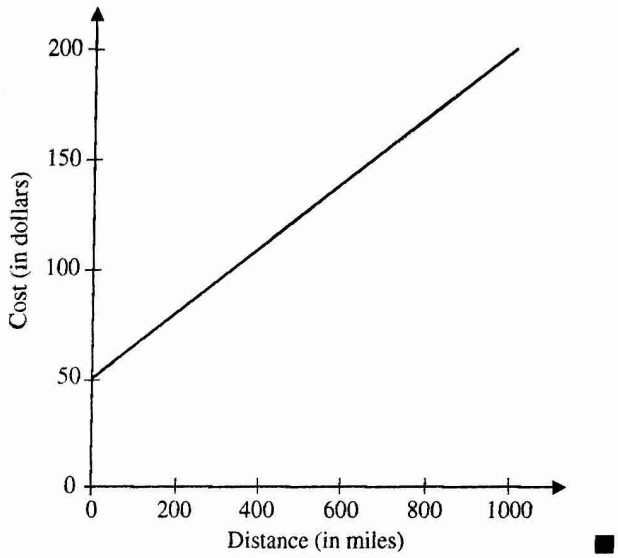
b) To find the graph of C as a function of d, it is useful to make a table of values of d and the corresponding values of C, like the following:

d (in miles)	C (in dollars)
0	50
50	57.50
100	65
200	80
300	95
500	125
1000	200

Each line of the table yields a point in the graph. For example, the first line goes to the point (0, 50) in the d-C plane. Placing the point into the plane is called **plotting** the point. By plotting all of the points in this table we obtain the following graph:



This gives you an incomplete graph, of course, because what if you drive 700 miles or 2000 miles? Well, you can plot those points, too. Obviously you'll never finish the complete graph by only plotting points. Can you just draw a smooth curve through the points to get the complete graph? In this case, yes, but in some other cases, no. One of the strengths of calculus is that it tells you whether drawing such a curve to get the complete graph is justified in any given case. In any event, using calculus we can show that in this case it is justified, and so we get the following:



■

Let's get a little more formal.

DEFINITION

A **function** f defined from a set A to a set B is a rule that associates with each element of the set A one, and only one, element of the set B.

For example, the expression $x^2 + 2x - 1$ represents a function defined on the set of all real numbers, and it associates the x value of 5 with the $x^2 + 2x - 1$ value of $25 + 10 - 1 = 34$. It also pairs the x value of 7 with the number 62, and so on. If we let $g(x)$ stand for $x^2 + 2x - 1$, that is, if $g(x) = x^2 + 2x - 1$, we can write $g(5) = 34$, $g(7) = 62$, $g(0) = -1$, $g(\sqrt{2}) = 2 + 2\sqrt{2} - 1 = 1 + 2\sqrt{2}$, $g(-1) = 12221522$, $g(a + b) = (a + b)^2 + 2(a + b) - 1$, etc.

In the definition above, A is called the **domain** of the function, and the set of all possible values of f is called the **range** of f. If the domain is not specified, we always take it to be **maximal**—as big as possible.

EXAMPLE 4 Suppose $f(x) = 2x^2$; what is $f(4), f(-3), f(4+h), f(x+\Delta x)$, and $f\left(\sqrt{\frac{x}{2}}\right)$?

(Think of Δx as a variable that happens to look a little different. Get used to it!)

Solution Simply replace x in the expression by the appropriate number. So

$$\begin{aligned} f(4) &= 2 \cdot 4^2 = 2 \cdot 16 = 32, \\ f(-3) &= 2 \cdot (-3)^2 = 2 \cdot 9 = 18, \\ f(4+h) &= 2(4+h)^2 = 2(16 + 8h + h^2) = 32 + 16h + 2h^2, \\ f(x+\Delta x) &= 2(x+\Delta x)^2 = 2(x^2 + 2x\,\Delta x + (\Delta x)^2) \\ &= 2x^2 + 4x\,\Delta x + 2(\Delta x)^2 \end{aligned}$$

and $$f\left(\sqrt{\frac{x}{2}}\right) = 2\left(\sqrt{\frac{x}{2}}\right)^2 = 2 \cdot \frac{x}{2} = x. \quad \blacksquare$$

EXAMPLE 5 The **absolute value function** is defined as

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

This is an important example of a function that is defined piecewise. It represents the distance on the number line from the number x to the origin. Even though "$-x$" may **look** negative, it is **not** negative if $x < 0$. For example, if $x = -5$, then $-x = -(-5) = 5$.

Also notice that $\sqrt{x^2} = |x|$. (Verify this for $x = 3$, and $x = -3$.) $\blacksquare$

Remark For any function f, we always have the following:

a) $f(x) > 0$ wherever the graph of $f(x)$ is above the x-axis.
b) $f(x) = 0$ wherever the graph of $f(x)$ crosses or touches the x-axis.
c) $f(x) < 0$ wherever the graph of $f(x)$ is below the x-axis.

4.1 Exercises

1) Let $f(x) = x^3 + 2x$. Determine:

a) $f(2)$ b) $f(3)$ c) $f(x+h)$

d) $f(2x)$ e) $f(-x)$ f) $f(2+\Delta x)$

2) Let $u(t) = \frac{1}{2}t^2 + t$. Determine:

   a) $u(2)$ b) $u(-1)$ c) $u(t + \Delta t)$

   d) $u(-t)$

3) a) Evaluate $|x|$ at $x = 0, \pm 1, \pm 2$, and sketch the graph.

   b) Indicate the parts of the number line where $|x|$ is greater than 2.

   c) Find the domain and range of $|x|$.

4) Evaluate:

   a) $|3 - 4|$ b) $|4 - 3|$ c) $|x^2 - x - 3|$ if $x = 2$

5) Solve $x^2 - 9 > 0$. (*Hint:* Plot points to sketch $x^2 - 9$, and see where the graph is above the x-axis.)

6) Solve $x^2 - 4 < 0$. (*Hint:* Plot points to sketch $x^2 - 4$, and see where the graph is below the x-axis.)

7) Let $f(x) = \sqrt{x} + 1$.

   a) What is its domain? b) Plot a few points to sketch a rough graph.

   c) What is its range?

8) Let $f(x) = \sqrt{x + 2} - 3$.

   a) What is its domain? b) Plot a few points to sketch a rough graph.

   c) What is its range?

9) Let $g(x) = x^2 + 2$.

   a) Plot a few points to sketch a rough graph. b) What are its domain and range?

10) It is estimated that 9 out of every 1 million \$20 bills are counterfeit. Design both a numerical (tabular) and symbolic representation that gives the predicted number of counterfeit \$20 bills in a sample of x million \$20 bills for $x = 0, 1, 2, 3, 4, 5, 6$.

11) The following table gives the average precipitation in Reno, Nevada, where $x = 1$ corresponds to January and $x = 12$ to December.

x (month)	1	2	3	4	5	6	7	8	9	10	11	12
P (in.)	0.5	0.3	0.4	0.3	0.2	0.1	0.3	0.5	0.2	0.4	0.4	0.3

   a) Determine the value of P during April and June.

   b) Is P a function of x? Explain.

   c) If $P = 0.3$, determine x.

4.2 Lines and Their Equations

Straight lines are the simplest of all curves, and one of the main ideas of calculus is to use lines to approximate complicated curves. So you've got to be proficient at lines and their equations. Lines have various degrees of steepness, which in mathematics is called **slope.** Consider Figure 4.1.

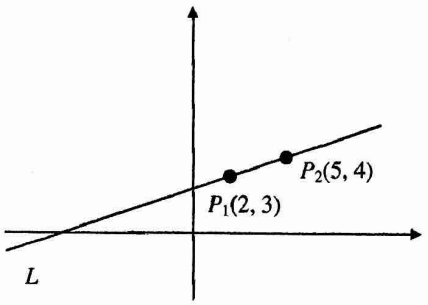


Figure 4.1

As you go from point P_1 to point P_2, you increase vertically a distance of 1 (called the **rise**) and horizontally a distance of 3 (called the **run**). The slope of line L is defined as $\frac{\text{rise}}{\text{run}} = \frac{1}{3}$.

Suppose the line L contains the two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ (see Figure 4.2):

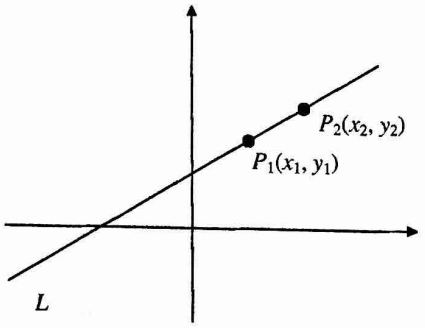


Figure 4.2

then L is said to have slope $m = \dfrac{y_2 - y_1}{x_2 - x_1}$, if $x_2 - x_1 \neq 0$.

Remarks

a) Notice that if $x_2 - x_1 = 0$, then $x_2 = x_1$ and the line is vertical, and hence we have defined the notion of slope only for nonvertical lines.

b) Notice also what happens if we choose not P_1 and P_2 but any two other points on L. By using similarity of triangles we can prove that the slope calculated using any two points on L is equal to the slope obtained using P_1 and P_2.

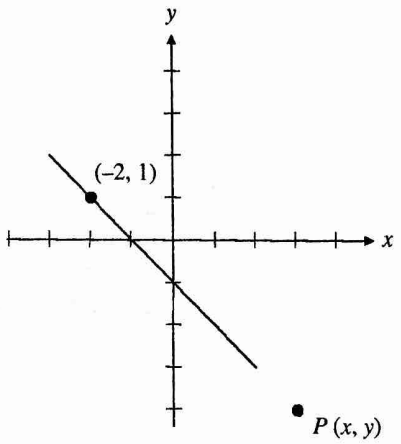
We have talked about graphs of functions. What about graphs of equations? Here's the idea.

DEFINITION

> The **graph of an equation** in x and y is the set of points (a, b) whose coordinates satisfy the equation when a is substituted for x and b is substituted for y.

EXAMPLE 1 Consider the line of slope -1 through the point $(-2, 1)$. Find an equation whose graph is that line. Equivalently: What is the equation of the line of slope -1 through $(-2, 1)$?

Solution Consider this graph:



Does $P(x, y)$ lie on the line? It does if, and only if, the slope is correct, that is, if $\frac{\text{rise}}{\text{run}} = -1$, meaning

$$\frac{y - 1}{x - (-2)} = -1.$$

We get rid of the denominator by multiplying by it, and then get

$$y - 1 = (-1)(x + 2).$$

So that is the equation of the line. ■

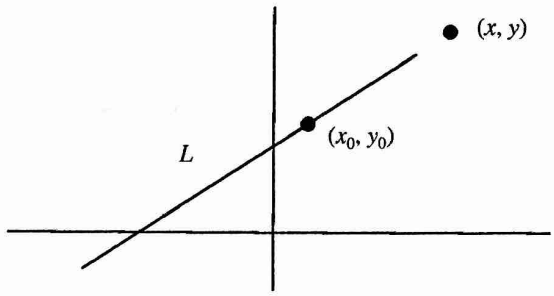
Theorem

The equation of the line of slope m going through the point (x_0, y_0) is

$$\boxed{y - y_0 = m(x - x_0)}.$$

Proof

We'll use the same idea as in the last example.



The point (x, y) lies on the line L if, and only if, the slope calculated from (x, y) and (x_0, y_0) is correct, that is: m. In other words $\dfrac{y - y_0}{x - x_0} = m$, which means $y - y_0 = m(x - x_0)$. (Notice that x and y are the variables, while m, x_0, and y_0 are just numbers.) This is called the **point-slope form** of the equation of a line. It is very important! Memorize it! (Right now is an excellent time.)

EXAMPLE 2 Find the equation of the line L of slope -3 going through the point $(2, -5)$.

Solution Nothing to it if you've memorized that equation. In this case, $m = -3$, $(x_0, y_0) = (2, -5)$, and x and y are just the variables in the equation.

So $$y - (-5) = -3(x - 2),$$
which cleans up to be $$y + 5 = -3(x - 2).$$

It's easy, once you know the slope and a point on the line. ■

EXAMPLE 3 Consider the lines through the origin of slope $m = 1, 2, -3$. Find their equations and graph them.

Solution We'll use the form $y - y_0 = m(x - x_0)$. In our case $(x_0, y_0) = (0, 0)$,

and so $$y - 0 = m(x - 0),$$

which simplifies to $$y = mx.$$

So the three given lines have equations $y = x$, $y = 2x$, and $y = -3x$. Their graphs are shown in Figures 4.3(a), 4.3(b), and 4.3(c), respectively. We can get the graphs by plotting merely one point in addition to the origin. Go 1 to the right, and up m, if $m > 0$, or down $|m|$ if $m < 0$.

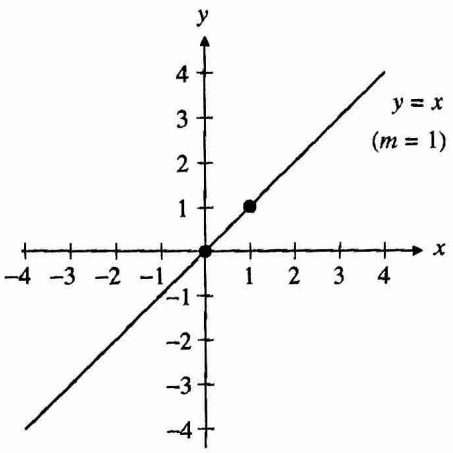


Figure 4.3(a)

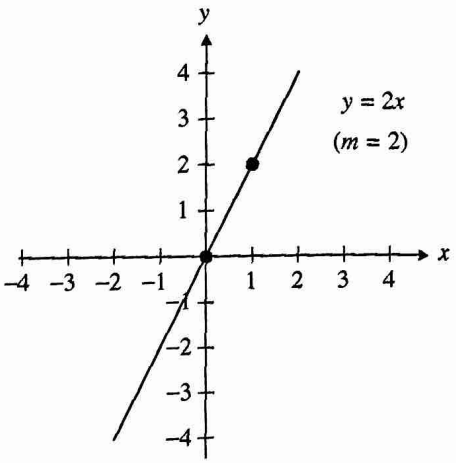


Figure 4.3(b)

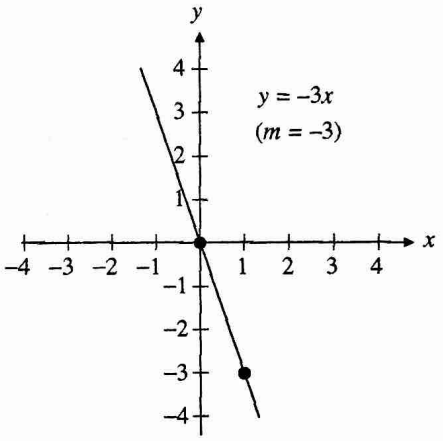
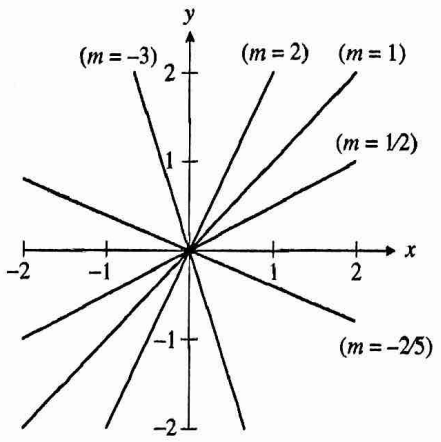


Figure 4.3(c)

■

Pictured in Figure 4.4 is the **family** of functions of the form $y = mx$, for several m, all on the same set of axes for comparison purposes. As we go from left to right (our usual assumption), if m is positive, the line rises; if it is negative, the line sinks. The larger the absolute value of the slope, $|m|$, the steeper the line becomes. (What happens when the slope is a really big positive number?)



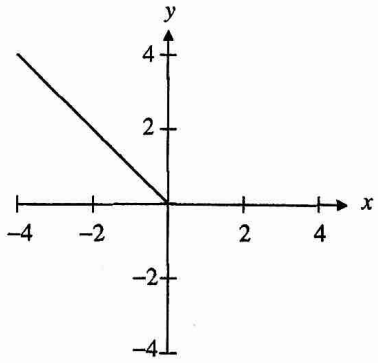
$y = mx$, for several m.

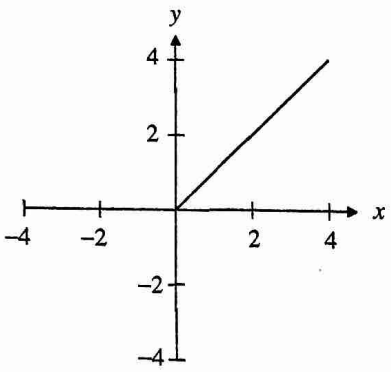
Figure 4.4

Remark It helps, whenever possible, to use the same scale on the x- and y-axes, meaning the distance from 0 to 1 is the same on the two axes. Sometimes that is really not feasible. (See Exercise 5.) In any event, always mark your scales—that is, show where 1, 2, etc. (or other appropriate values) are.

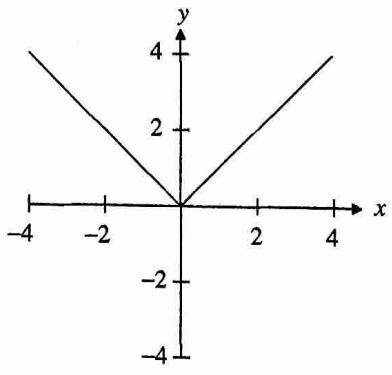
EXAMPLE 4 Graph $h(x) = |x|$.

Solution If $x < 0, h(x) = -x$. If $x \geq 0, h(x) = x$.





Put it together, and we get $h(x) = |x|$.



■

All nonvertical lines cross the y-axis. If a line crosses the y-axis at the point $(0, b)$, then b is called the **y-intercept**. To find the equation of the line of slope m and y-intercept b is easy. We use the point-slope form of the equation of a line, where the slope is m, and the line contains the point $(0, b)$. Doing so we get $y - b = m(x - 0)$, which reduces to

$$\boxed{y = mx + b}.$$

This is called the **slope-intercept form**. You should also memorize this.

EXAMPLE 5 Consider the equation $6x - 3y = 15$. What is its graph?

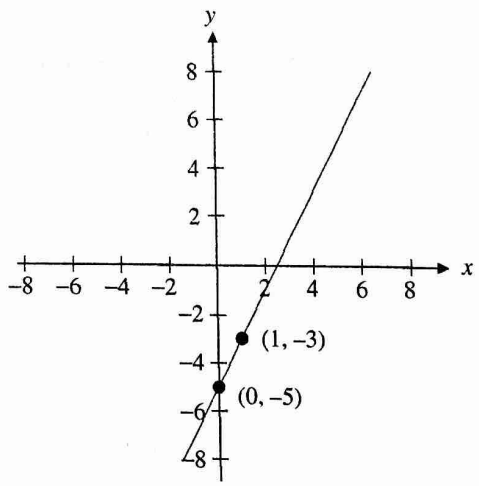
Solution We could make a table by putting $x = 1, 2, 3$, etc., and calculating the corresponding y-values, and then plotting points, but we won't. Instead, we solve for y:

$$-3y = -6x + 15$$

or

$$y = 2x - 5,$$

which we know is the equation of a line of slope 2 going through $(0, -5)$. If we start from $(0, -5)$ and go over 1 and up m (in this case $m = 2$), we get a second point on the line $(1, -3)$, and hence we know the whole graph.



■

Lastly, we remind you that parallel lines have equal slopes, and perpendicular lines have slopes that are negative reciprocals of each other. This means, if line L_1 has slope m_1, and line L_2 has slope m_2, then L_1 is parallel to L_2 if, and only if, $m_1 = m_2$; moreover, L_1 is perpendicular to L_2 if, and only if, $m_2 = \frac{-1}{m_1}$. Consider the following.

EXAMPLE 6 Find the equation of the line passing through the point (1, 3) that is parallel to the line $2x + 3y = 6$.

Solution First, we determine the slope of the given line by putting it in slope-intercept form. We solve $2x + 3y = 6$ for y to get:

$$y = -\frac{2}{3}x + 2.$$

Comparing this to $y = mx + b$, we see that the slope of the given line is $-\frac{2}{3}$. Since the point on the desired line is (1, 3), we use the point-slope form to get

$$y - 3 = -\frac{2}{3}(x - 1),$$

which is one form of the answer. Depending on your purpose, you may wish to change it to slope-intercept form, or leave it as is. ■

EXAMPLE 7 Find the equation of the line passing through the point $(3, -2)$ that is perpendicular to the line $y = 6x + 4$.

Solution The line $y = 6x + 4$ has slope equal to 6. Any line perpendicular to this line will have slope equal to $-\frac{1}{6}$. (This is the negative reciprocal of 6, right?) So the line we want passes through the point $(3, -2)$ and has slope $m = -\frac{1}{6}$. Using the point-slope form, we obtain:

$$y - (-2) = -\frac{1}{6}(x - 3)$$

or

$$y = -\frac{1}{6}x - \frac{3}{2}.$$ ■

4.2 Exercises

In Exercises 1–5, graph the function by plotting two points each. (Why is it enough to plot only two points?)

1) $f(x) = \frac{1}{3}x$

2) $y = -1.5x$

3) $g(x) = 0.6x$

4) $f(x) = 5x$

5) $y = 1000x$

6) Find the equation of the line of slope $\frac{2}{3}$ through the point (2, 7).

7) Find the equation of the line of slope -6 through the point $(-3, 2)$.

8) Find the equation of the line containing the two points $(1, 2)$ and $(-2, 1)$. (*Hint:* Find m.)

9) Find the equation of the line containing the two points $(-1, 4)$ and $(2, 1)$.

10) Graph the line given by the equation $4y - 3x = 24$.

11) Find the point-slope form for the equation of the line through the point $(1, 4)$ parallel to the line $y = 2x - 5$. Graph the line.

12) Find the point-slope form for the equation of the line L through the point $(4, 2)$ parallel to the line $y = 9x - 5$. Graph the line L.

13) Find the slope-intercept form for the equation of the line through the point $(-1, 4)$ perpendicular to the line $y = \frac{1}{3}x - 5$. Graph this line, and label the y-intercept.

14) Find the slope-intercept form of the equation of the line L through the point $(1.34, 2.67)$ perpendicular to the line $y = -0.235x - 5.29$. Graph this line L, and label the y-intercept.

15) Find the equation of the line through the point $\left(0, \frac{4}{3}\right)$ parallel to the line $y = 5x - 10$. Graph both lines.

16) Graph the line given by the equation $4.014y - 2.276x = 9.87$.

17) Find the equation of the line through the point $\left(\frac{1}{2}, -3\right)$ perpendicular to the line $y = -\frac{1}{7}x - 5$. Graph both lines.

18) Find the equation of the line through $(3, -4)$ parallel to the line $2x - y = 5$, and put it into slope-intercept form.

19) The temperature measured in degrees Fahrenheit, F, is a linear function of the temperature measured in degrees Celsius, C. The ordered pair, or point, $(0, 32)$ is an ordered pair of this function because 0°C is equivalent to 32°F, the freezing point of water. The ordered pair $(100, 212)$ is also an ordered pair of this function since 100°C is equivalent to 212°F, the boiling point of water. a) Use these two points to write F as a function of C, and graph it. b) Find the temperature in degrees Fahrenheit of a thermometer reading 45°C.

20) In statistics, one way to analyze data is by determining the "line of best fit," also called a **regression line**. Once this regression line is calculated, one may use it to predict data values in between the collected data. Given a set of data points $\{(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots, (x_n, y_n)\}$, the regression line is given by $y = ax + b$, where a and b are functions of the actual data, x_i and y_i, for $i = 1$ to n. (The exact formulas are too complicated to give here right now.) Suppose a particular set of data gives rise to values $a = 1.078$ and $b = 3.4457$. Graph this particular line of regression. What is y if $x = 2.34$? What is the significance of the y-intercept?

4.3 Power Functions

Functions of the form $f(x) = x^r$, where r is a constant, are called **power functions.** Let's look at their graphs.

Part A: $y = x^r$, with $r = 2, 4, 6$, etc. (See Figure 4.5.)

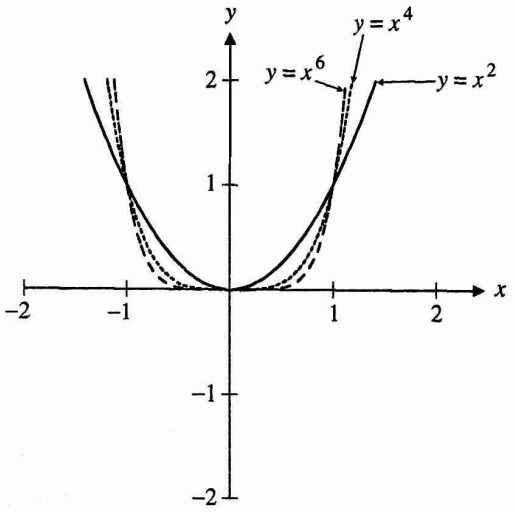


Figure 4.5

Notice how the points $x = 1$ and $x = -1$ are "pivotal"? Consider $x > 0$, or the right side of the graph. For $x < 1$, the relative position of the curves is opposite to what it is when $x > 1$. Well, let's think about it: For $x < 1$, the more we multiply it by itself, the smaller it gets. The higher the power, the smaller the number gets. When $x > 1$, the opposite is true. The higher the power, the more we multiply by a number that is greater than 1, hence the larger the number gets. Notice that these graphs seem symmetric about the y-axis. In fact, they are. It doesn't matter whether we insert x or $-x$ into an even-powered function; the result is the same. (Can you show this?) This shows **symmetry about the y-axis.**

Part B: $y = x^r$, with $r = 3, 5, 7$, etc.

Symmetry through the origin is exhibited by the odd-powered functions. Figure 4.6 shows three members of this family.

As in Figure 4.5, the points $x = 1$ and $x = -1$ are pivotal in that the relative positioning of the graphs changes there. Now, however, the graphs are symmetric through the origin. That is, if the point (x, y) is on the graph, then so is $(-x, -y)$.

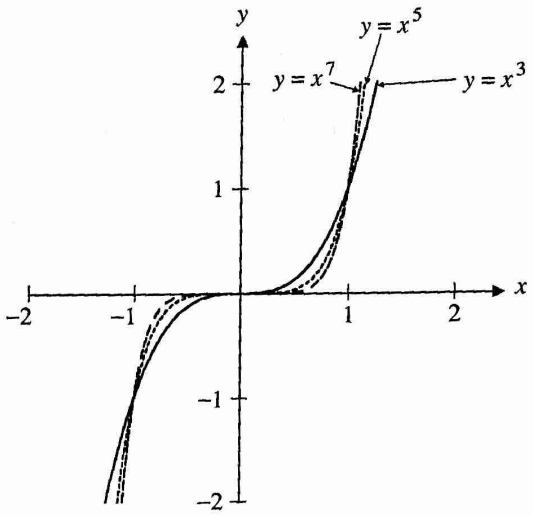


Figure 4.6

Part C: $y = x^r$, with $r = \frac{1}{2}, \frac{1}{4}, \frac{1}{6}$, etc., and $r = \frac{1}{3}, \frac{1}{5}, \frac{1}{7}$, etc.

What if r is a positive fraction of the form $\frac{1}{n}$, where n is a whole number 1, 2, 3, etc.? What do $y = x^{1/2}$, $y = x^{1/3}$, etc. look like? Well, as it turns out, this family of graphs also splits up into two groups, namely, the even n graphs and odd n graphs. Within each group, the graphs display similar behavior. For $y = x^{1/2}$ and $x^{1/4}$, the graphs look like that in Figure 4.7. Since $y = x^{1/n}$ is equivalent to the nth root, if n is even, x must not be negative.

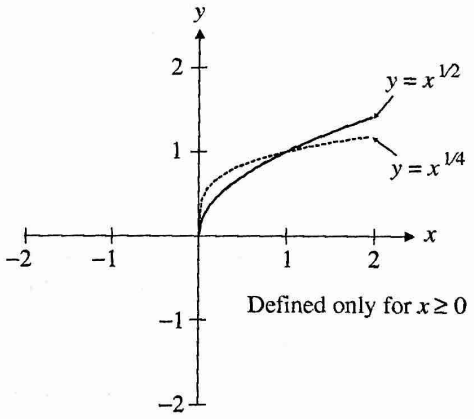


Figure 4.7

The other graphs of $y = x^{1/n}$ for n even are very similar to those shown in Figure 4.7, and as is the case in Figures 4.5 and 4.6, the point $x = 1$ is where the relative positioning of the graphs changes. If n is odd—say, for example, $y = x^{1/3}$ or $y = x^{1/5}$—then x can be any real number. The graphs of these functions are shown in Figure 4.8.

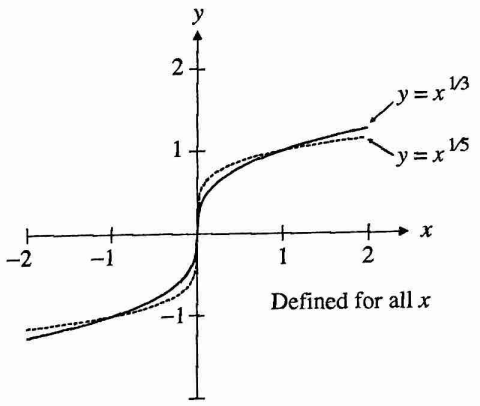


Figure 4.8

The other graphs of $y = x^{1/n}$ for n odd are very similar to those shown in Figure 4.8, with the same ordering change at $x = 1$ and $x = -1$.

To sum up Parts A, B, and C: For $x \geq 0$, $y = x^r$ is shown in Figure 4.9.

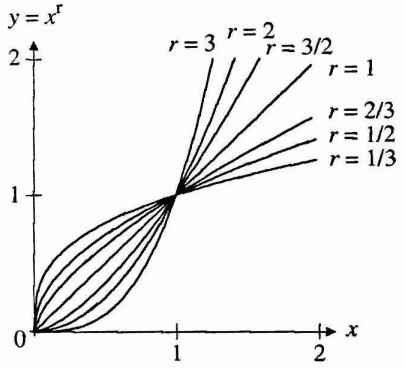


Figure 4.9

All of these are examples of **power functions**. Notice that $x^{2/3}$ is calculated by squaring $x^{1/3}$.

Part D: $y = x^r$, with $r = -1, -3$, etc., and $-2, -4$, etc.

What if r is a negative integer? Can you guess that the situation will divide up into two cases, r even or odd? Figure 4.10 shows the situation for r odd: $\frac{1}{x}, \frac{1}{x^3}$, etc. (Recall that x^{-3} means $\frac{1}{x^3}$, etc.)

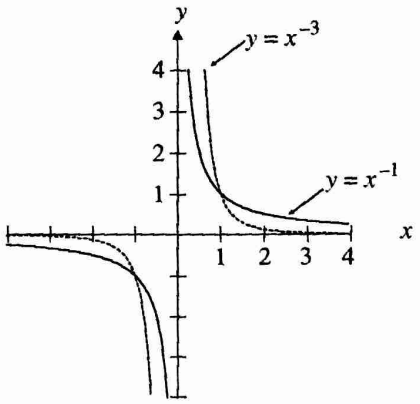


Figure 4.10

For $r = -2$ and -4, the graph is shown in Figure 4.11.

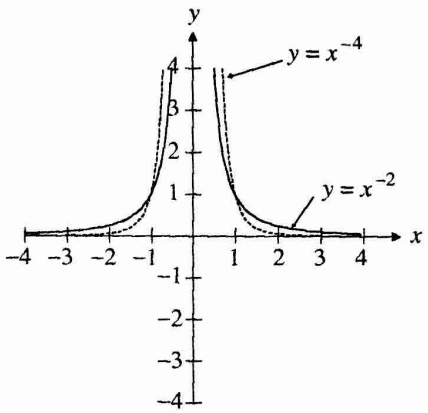


Figure 4.11

The functions x^{-3}, x^{-5}, etc. all basically look like x^{-1}, and x^{-4}, x^{-6}, etc. basically look like x^{-2}. The points $x = 1$ and $x = -1$ are again important since the relative positioning changes there.

4.3 Exercises

Using graph paper, calculate and plot exactly at least four points each for Exercises 1–9.

1) $f(x) = \sqrt{x}$

2) $f(x) = x^3$

3) $g(x) = \sqrt[3]{x}$

4) $f(x) = \dfrac{1}{x}$

5) $g(x) = \dfrac{1}{x^2}$

6) $w(x) = x^{-3}$

7) $f(x) = x^{2/3}$

8) $f(x) = x^{3/2}$

9) $f(x) = \dfrac{1}{\sqrt{x}}$

10) Using graph paper, calculate and plot at least six points for $f(x) = x^2$. Now sketch a curve through these points as an approximation to the graph of f. Similarly, on the same graph paper, sketch $g_1(x) = 2x^2$, $g_2(x) = \frac{1}{2}x^2$, $g_3(x) = -2x^2$, and $g_4(x) = -\frac{1}{2}x^2$. How are these four functions related to the original function f?

11) Sketch $y = ax^n$, where n is an integer greater than 2 and a is real. Consider the cases for n even or n odd separately, and plot the graphs for $a > 0$ or $a < 0$ on the same axes.



Shifting Up or Down

How does the graph of $y = x^2 + 2$ compare to the graph of $y = x^2$? Well, all the y-coordinates of the first graph are 2 bigger than those of the second graph. But y is the altitude, or height, of the point (x, y). So to go from the graph of $y = x^2$ to $y = x^2 + 2$, just push up the graph a distance of 2, as shown in Figure 4.12.

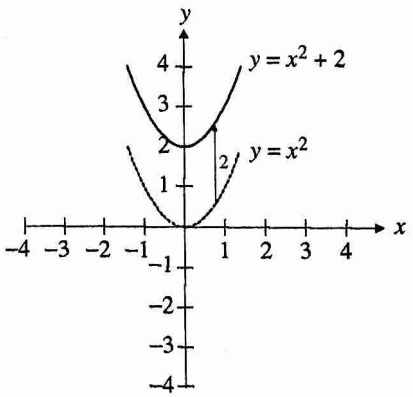


Figure 4.12

Also consider the examples in Figures 4.13 and 4.14.

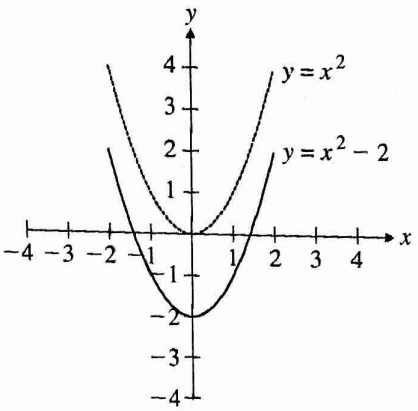


Figure 4.13(a)

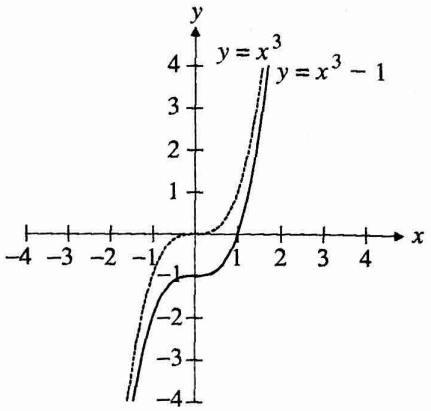


Figure 4.13(b)

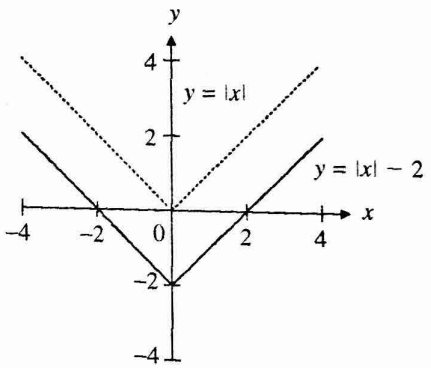


Figure 4.14

In each case, we are graphing a function by using the fact that it is shifted up or down from the graph of a function that we know.

4.4 Exercises

1) Graph the following functions:

a) $y = \frac{1}{2}x$ b) $y = \frac{1}{2}x + 2$ c) $y = \frac{1}{2}x - 1$

2) Graph the following functions:

a) $y = \frac{1}{2}x^2$ b) $y = \frac{1}{2}x^2 + 1$ c) $y = \frac{1}{2}x^2 - 2$

3) Graph the function $y = -2x + 3$.

4) Graph the function $y = x^3 + 5$.

5) Graph the following functions:

a) $y = x^2 - 4$ b) $y = x^2 + \pi$ c) $y = x^{-2} + 1$

6) Graph the following functions:

a) $y = x^2 + 2$ b) $y = x^{-2} - 2$

7) Graph the function $y = \sqrt{x} + 2$ for $x \geq 0$.

8) Graph $|x|$ and $|x| + 1$.

9) Let $f(x) = \frac{1}{x}$. Graph $f(x)$ and $f(x) + 1$.

10) Let $g(x) = \frac{1}{x^2}$. Graph $-g(x)$ and $2 - g(x)$.

4.5 Shifting Left or Right

Consider the three graphs in Figure 4.15.

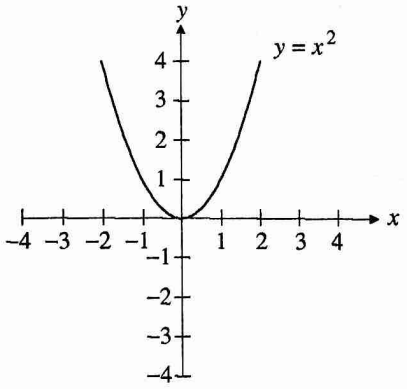


Figure 4.15(a)

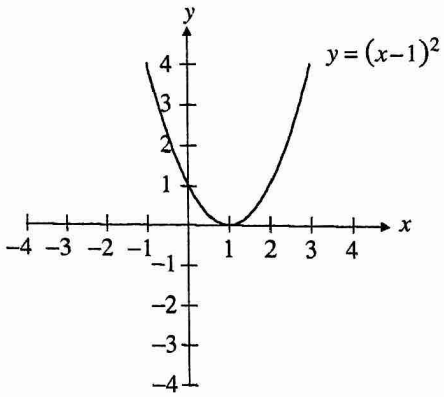


Figure 4.15(b)

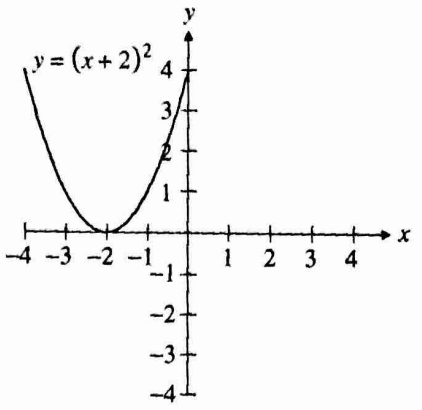


Figure 4.15(c)

(Don't take **our** word for it; test these by plotting points for $x = 0, \pm 1, \pm 2, \pm 3$.)

Notice that going from $f(x) = x^2$ to $f(x - 1) = (x - 1)^2$ we needed to shift the graph to the **right** a distance of 1, and going from $f(x) = x^2$ to $f(x + 2) = (x + 2)^2$ shifted the graph to the **left** a distance of 2. **This is true for all functions $f(x)$**, not just $f(x) = x^2$. In general, if we suppose a is some positive number, then the graph of $f(x - a)$ is the graph of $f(x)$ moved to the right a distance of a, and the graph of $f(x + a)$ is the graph of $f(x)$ moved to the left a distance of a. For instance, check the graphs in Figure 4.16.

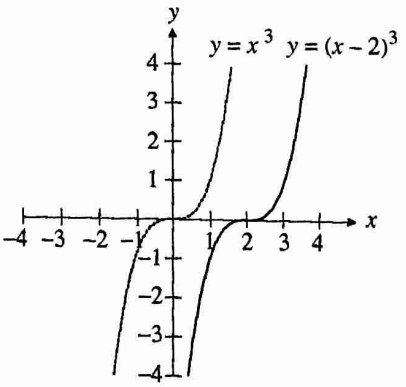


Figure 4.16(a)

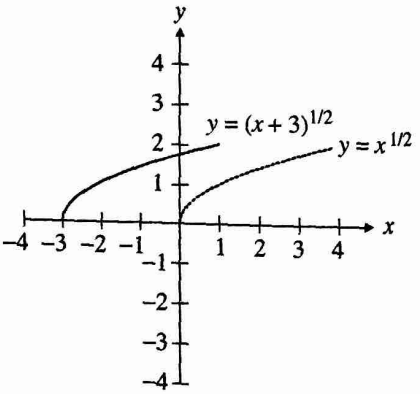


Figure 4.16(b)

Remarks

a) Notice in Figures 4.15(c) and 4.15(b) that the "vertex" of $(x + 2)^2$ is at -2, while the vertex of $(x - 1)^2$ is at $+1$, which may be opposite to what you might have expected, but plotting a few points proves it.

b) It is important to see that, for example, $(x + 3)^2$ is very different from $x^2 + 3$. If we start from the graph of x^2, $(x + 3)^2$ moves it **3 units to the left,** while $x^2 + 3$ moves it **up a distance of 3 units.**

4.5 Exercises

1) Graph x^2, $(x + 1)^2$, and $(x - 1)^2$, by plotting five points for each.

2) Graph the following functions:

a) $f(x) = (x - 3)^2$ b) $y = (x + \pi)^2$ c) $y = \dfrac{1}{x - 1}$

3) Graph the following functions:

a) $y = (x + 3)^3$ b) $y = \sqrt[3]{x + 1}$ c) $y = -\dfrac{1}{(x - a)^4}$, where $a > 0$

4) Graph the function $y = \sqrt{x - 4}$ for $x \geq 4$.

5) Graph $g(x) = |x|$ and $h(x) = |x - 3|$.

6) Graph the function $y = \sqrt{x + 2}$ for $x \geq -2$.

7) Let $B(t)$ be the number of births during year t in the United States. Assume for the moment that 60% of these people will graduate from high school at the age of 18. Let $G(t)$ be the number of graduates from high school during year t in the United States. What approximate relation would you expect between the functions $B(t)$ and $G(t)$? How could this be used by college admissions staff members?

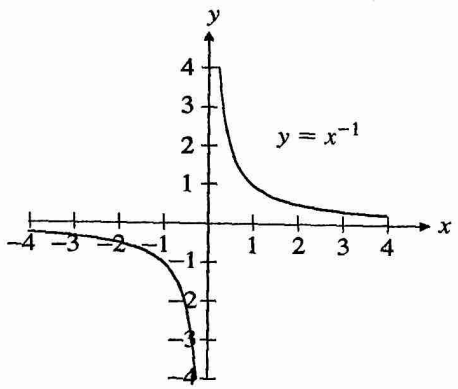
4.6 Translations

We can put all this shifting together to get functions all over the place. Such combinations of vertical and horizontal shifts are called **translations**.

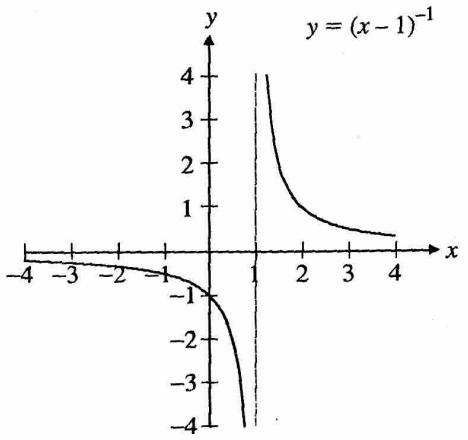
EXAMPLE 1 Graph the function $y = \dfrac{1}{(x - 1)} + 2$.

Solution We graph this by a) starting with the graph for $y = \frac{1}{x}$, b) shifting this right 1 unit to get $y = \frac{1}{(x - 1)}$, and then c) shifting up 2 to get the final graph $y = \frac{1}{(x - 1)} + 2$.

a)

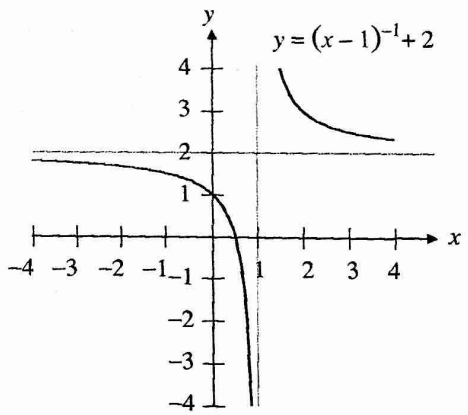


b)



Notice that the vertical asymptote at $x = 0$ moves over to $x = 1$. (An **asymptote** is a straight line that a curve approaches closer and closer. The rigorous definition will be given in your calculus book, because it involves the notion of limit.)

c)



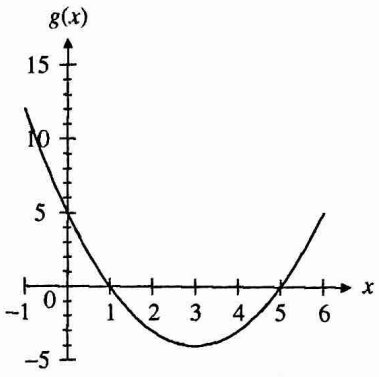
Notice that the horizontal asymptote at $y = 0$ moves up to $y = 2$. ■

EXAMPLE 2 Graph $g(x) = x^2 - 6x + 5$.

Solution The trick is to complete the square.

$$g(x) = (x^2 - 6x + 9) + (5 - 9)$$
$$= (x - 3)^2 - 4$$

So, $g(x)$ looks like x^2 but translated 3 units right and 4 units down.



■

4.6 Exercises

Graph the following functions:

1) a) $y = x^2$ b) $y = x^2 - 1$

c) $y = \frac{1}{2}(x^2 - 1)$ d) $y = \frac{1}{2}(x^2 - 1) + 2$

2) a) $y = x^2$ b) $y = x^2 - 3$

c) $y = (x - 2)^2 - 3$

3) a) $y = \frac{1}{x}$ b) $y = \frac{1}{x - 2}$

c) $y = \frac{3}{x - 2}$ d) $y = \frac{3}{x - 2} - 1$

4) a) $y = \frac{1}{x^2}$ b) $y = \frac{1}{(x + 1)^2}$

c) $y = \frac{1}{(x + 1)^2} + 4$

5) a) $y = \sqrt{x}$ b) $y = \sqrt{-x}$

c) $y = \sqrt{-(x - 2)} = \sqrt{2 - x}$ d) $y = 1 + \sqrt{2 - x}$

6) $y = (x - 1)^2 - 1$

7) $y = \frac{1}{x + 1} + 2$

8) $y = (x - 2.34)^3 - 7.21$

$y = 2 - \sqrt{x - 2}$

$y = \dfrac{1}{x - 1} + 5$

$y = (x - 1)^3 + 2$

$y = 1 - \dfrac{1}{(x - \pi)^4}$

4.7 Intersection of Curves and Simultaneous Solutions

We know that a point (a, b) lies on a curve in the xy-plane if, and only if, $x = a$ and $y = b$ are solutions of the equation of that curve. (For example, the curve $y = x^2$ has the point $(2, 4)$ on it because $2^2 = 4$.) What about a point of intersection of two curves? Well, that point lies on both curves, so its coordinates must satisfy the equations of both curves at the same time! That is, x and y must satisfy the two equations **simultaneously.** A set of two (or more) equations of two (or more) variables is called a **system of equations.** Finding solutions of such systems of equations is usually easy.

EXAMPLE 1 Find the intersection points of the curves whose equations are $y = x^2 - 4$ and $x + y = 8$.

Solution Since both equations have to be true at the same time, we can solve for either x or y in one equation and substitute that result into the other equation. Here, let's solve for y in the first equation (it's already done!) and substitute that into the second equation, to give us

$$x + (x^2 - 4) = 8.$$

We clean this up and put it into standard form:

$$x^2 + x - 12 = 0.$$

Here we can either use the quadratic formula or factor it to get

$$(x + 4)(x - 3) = 0.$$

By the Zero-Factor Property, $x = 3$ or -4. (See Section 3.3 for more on the ZFP.) So we know that there are two points of intersection: one where $x = 3$ and the other where $x = -4$. To find the corresponding y-values, you can substitute each x-value into either of the two equations (pick the easier one) to calculate y.

So: $$y = x^2 - 4,$$

hence if $x = 3$, $y = 3^2 - 4 = 5$, and if $x = -4$, $y = (-4)^2 - 4 = 12$. Therefore, the two points of intersection are $(3, 5)$ and $(-4, 12)$. ■

EXAMPLE 2 Where do the lines

$$2x + 3y = 7$$

and

$$-3x + y = 11$$

intersect?

Solution Here, let's start with the second equation to get $y = 3x + 11$ and substitute that into the first equation to get

$$2x + 3(3x + 11) = 7.$$

We clean this up and solve:

$$2x + 9x + 33 = 7$$

$$11x = -26$$

or

$$x = \frac{-26}{11} = -\frac{26}{11}.$$

To find the corresponding y-value, use $y = 3x + 11$ with $x = -\frac{26}{11}$ in it to get

$$y = 3\left(-\frac{26}{11}\right) + 11$$

or

$$y = -\frac{78}{11} + 11 = \frac{-78 + 121}{11} = \frac{43}{11}.$$

Hence there is one intersection point: $\left(-\frac{26}{11}, \frac{43}{11}\right)$. ■

Remark The method we used in these two examples is called the **method of substitution.** If all the equations are of degree 1 in whatever variables there are (as in Example 2), there is another way we can write the system using matrices. You can then use methods from the mathematical subject of linear algebra to easily solve such problems.

4.7 Exercises

1) Find the simultaneous solutions to the following systems of equations:

a) $\begin{cases} 3x - 2y = 16 \\ 5x + y = 5 \end{cases}$ b) $\begin{cases} x^2 - 4y = 6 \\ 2x + 2y = 3 \end{cases}$

2) Find the intersection of the curves:

a) $x^2 = y - 2$ and $2x + 3y = -7$ b) $x + y = 2$ and $\sqrt{x} + 4y = -6$

3) Find the points of intersection for the line $y = x$ and the circle $x^2 + y^2 = 1$. (*Hint*: Graph the functions.)

4) Find the intersection of the curves:

   a) $x^2 = 2.20y - 1$ and $2.01x + 3.2y = -2$

5) Find all the intersection points of the unit circle centered at the origin with the line $x = \frac{1}{2}$. Sketch.

6) Find the points of intersection for the line $y = -x$ and the circle $x^2 + y^2 = 1$. (*Hint:* Graph the functions.)

7) Sketch the circle of radius 2 centered at (1, 1). Find all points on this circle whose altitude (height above the x-axis) is 2.