CHAPTER

3

Solving Equations

3.1 Equations of Degree 1 *(Linear Equations)*

You're probably asking: Why are we doing this? This is the easiest thing in algebra. Well, it is, but it can get confusing when too many different variables are floating about. As long as you keep things straight, it really *is* easy. We'll start off simple, and each new example will bring up some new twist and how to deal with it.

The expression $ax + b$, with $a \neq 0$, is a polynomial of degree 1, and so the equation $ax + b = 0$ is called an **equation of degree 1**. Since the graph of the function $ax + b$ is a straight line (see Section 4.2), the equation $ax + b = 0$ is also called a **linear equation**.

EXAMPLE 1 Solve for x: $4x - 16 = 0$.

Solution First, get rid of the -16 by adding 16 to both sides:

$$4x - 16 + 16 = 0 + 16,$$

which gives $$4x = 16.$$

Now, get rid of the 4 in front of the x by dividing both sides by 4:

$$\frac{4x}{4} = \frac{16}{4},$$

giving $$x = 4.$$

Easy!

It's always a good idea to check your answer by substitution. That is, if we substitute $x = 4$ back into the original equation, it works! It is certainly true that

$$4 \cdot 4 - 16 = 0. \quad \blacksquare$$

EXAMPLE 2 Solve for x: $\frac{2}{3}x + 1 = 0.$

Solution Get rid of the 1 by subtracting it from both sides. This gives

$$\frac{2}{3}x + 1 - 1 = 0 - 1$$

or

$$\frac{2}{3}x = -1.$$

Now, to isolate the variable x we need to divide by $\frac{2}{3}$. So, just like in Chapter 1, we multiply by its reciprocal, $\frac{3}{2}$:

$$\frac{3}{2} \cdot \frac{2}{3}x = \frac{3}{2} \cdot (-1).$$

This gives

$$x = -\frac{3}{2}.$$ ■

EXAMPLE 3 Solve for y: $\sqrt{2}y - 4 + \sqrt{2} = 0.$

Solution While this looks a little more complicated, y appears in only one place, so you can start "peeling the onion." This is what we will call the method of stripping off terms, or factors, until the desired variable is exposed.

In this case, you first peel away the $-4 + \sqrt{2}$ by subtracting it from both sides to get

$$\sqrt{2}y = -(-4 + \sqrt{2})$$

or

$$\sqrt{2}y = 4 - \sqrt{2}.$$

The next layer to peel away is the $\sqrt{2}$ on the left side of the equation. Divide by $\sqrt{2}$ to get

$$y = \frac{4 - \sqrt{2}}{\sqrt{2}}.$$

We can simplify this answer as follows:

$$y = \frac{4 - \sqrt{2}}{\sqrt{2}} = \frac{4}{\sqrt{2}} - \frac{\sqrt{2}}{\sqrt{2}} = 2\sqrt{2} - 1.$$ ■

In each of the preceding examples we obtained a solution by peeling the onion. The steps taken in stripping away the layers to expose the desired variable involved addition, subtraction, multiplication, or division of real numbers. Things can get a little more confusing when the equation involves more than the one variable you are solving for, but the idea is exactly the same. So, keep in mind what you have learned, and forge ahead!

EXAMPLE 4 Solve for x: $y^2x + w^2 = 0$.

Solution Now the equation has several variables. Remember the variable you wish to solve for and concentrate on isolating that one variable. Here, we wish to solve for x. Even though things seem a little more complicated, x still appears in only one place, so you can start peeling. In this case, first peel away the w^2 by subtracting it from both sides to get

$$y^2x = 0 - w^2$$

or

$$y^2x = -w^2.$$

Next, peel off the y^2 by division:

$$\frac{y^2x}{y^2} = \frac{-w^2}{y^2},$$

resulting in

$$x = \frac{-w^2}{y^2}. \blacksquare$$

Again, we solve the equation by stripping away the layers to expose the desired variable. Unlike the first three examples, the last example also had variables, w and y, in addition to the variable we were trying to solve for, x. These variables, however, just stand for real numbers, and so the mathematical steps taken to peel them away are exactly the same as those taken in the other examples.

Sometimes the variable you wish to solve for occurs more than once. In many cases this is not a problem because you can rewrite the equation in such a way that the variable occurs only once, and then solve as usual.

EXAMPLE 5 Solve for x: $4x + 3 = 2x + 1$.

Solution Here x occurs more than once, but if we subtract $2x$ from both sides, we get

$$2x + 3 = 1.$$

Notice that we have, in effect, moved all the terms containing x to the left side of the equation. Now, we subtract 3 from both sides to get

$$2x = -2,$$

where all the terms not containing x are now moved to the right side. Dividing by 2 gives the final answer:

$$x = \frac{-2}{2} = -1. \blacksquare$$

EXAMPLE 6 Solve for x: $2x + 5y = 3x + y + 1$.

Solution Even though this appears more complicated because of the extra y-terms floating around, it really isn't any more difficult. Here, the x occurs more than once, so we subtract $3x$ from both sides, to get

$$-x + 5y = y + 1.$$

Notice that the terms containing x are all on the left side. Now, we subtract $5y$ from both sides:

$$-x = y + 1 - 5y,$$

which gives

$$-x = 1 - 4y.$$

Notice that the terms not containing x are all on the right side. Dividing by -1 gives the final answer:

$$\begin{aligned} x &= \frac{(1 - 4y)}{-1} \\ &= \frac{1}{-1} - \frac{4y}{-1} \\ &= -1 - (-4y) \\ &= -1 + 4y \\ &= 4y - 1. \quad \blacksquare \end{aligned}$$

EXAMPLE 7 Solve for y: $\frac{2}{3}y + 2x - 1 = \frac{3}{4}y + x - \frac{1}{2}$.

Solution Here y occurs more than once, but we can proceed as in the last example. First we subtract $2x - 1$ from both sides:

$$\frac{2}{3}y = \frac{3}{4}y + x - \frac{1}{2} - (2x - 1),$$

which is the same as

$$\frac{2}{3}y = \frac{3}{4}y + x - \frac{1}{2} - 2x + 1,$$

or

$$\frac{2}{3}y = \frac{3}{4}y + x - 2x + 1 - \frac{1}{2},$$

and finally

$$\frac{2}{3}y = \frac{3}{4}y - x + \frac{1}{2}.$$

Now, we can combine the two terms that contain y by subtracting $\frac{3}{4}y$ from both sides to get

$$\frac{2}{3}y - \frac{3}{4}y = \frac{3}{4}y - x + \frac{1}{2} - \frac{3}{4}y.$$

Canceling the $\frac{3}{4}y$'s on the right-hand side (RHS), and combining the y-terms on the left-hand side (LHS) gives:

$$\left(\frac{2}{3} - \frac{3}{4}\right)y = -x + \frac{1}{2}.$$

Remember how to subtract fractions: We get a common denominator, etc.

So $$\frac{2}{3} - \frac{3}{4} = \frac{2 \cdot 4}{12} - \frac{3 \cdot 3}{12} = \frac{8}{12} = \frac{9}{12} = -\frac{1}{12}$$

and hence $$-\frac{1}{12}y = -x + \frac{1}{2}.$$

Now, to expose the variable y, we divide by $-\frac{1}{12}$, that is, we multiply both sides by -12 to give the solution:

$$(-12) \cdot \left(-\frac{1}{12}\right)y = (-12) \cdot \left(-x + \frac{1}{2}\right)$$

or
$$\begin{aligned} y &= (-12) \cdot (-x) + (-12) \cdot \left(\frac{1}{2}\right) \\ &= 12x - 6. \quad \blacksquare \end{aligned}$$

EXAMPLE 8 You bought two ties, three shirts (each of which cost twice as much as a tie), and a sweater that cost twice as much as a shirt, spending a total of \$180. How much did the sweater cost?

Solution We let x = price of a tie and express the other quantities in terms of x. So the price of a shirt is $2x$, and the price of a sweater is $4x$. Next, we express the fact that the total is \$180 in the form of an equation, and solve it.

$$\begin{aligned} 2(x) + 3(2x) + 4x &= 180 \\ 2x + 6x + 4x &= 180 \\ 12x &= 180 \\ x &= 15 \end{aligned}$$

But, \$15 is not the answer to the question! (Read the question.) The sweater cost $4x = 60$ (i.e., \$60). ■

3.1 Exercises

1) Solve for x: $3x - 4 = 0$

2) Solve for x: $2.34x - 20.23 = 0$

3) Solve for x: $\frac{5}{12}x - 35 = 0$

4) Solve for z: $\frac{9}{2}z + \frac{1}{3} = \frac{1}{6}$

5) Solve for z: $\frac{4}{3}z - 1 = \frac{1}{10}$

6) Solve for x: $1.23x - 3.01 = 0.67x + 2.56$

7) Solve for x: $\frac{2}{3}x - 3 = \frac{1}{5}x + 2$

8) Solve for y: $\frac{1}{4}y - \frac{1}{8} = \frac{1}{16} - 2y$

9) Solve for y: $\frac{1}{2}y - \frac{1}{3} = \frac{1}{6} - 2y$

10) The total cost for renting a copy machine for 60 months at \$100 per month plus 5 cents a copy is $0.05x + 6000$ dollars, if x copies are made. If the same copy machine is purchased for \$2000, then it costs 7 cents per copy for supplies and maintenance, for a total cost of $2000 + 0.07x$ dollars for x copies. Solve the equation $0.05x + 6000 = 2000 + 0.07x$ to find the number of copies for which the 5-year cost is the same as purchasing. Assuming the copier has no value after 5 years and assuming you make 2000 copies per month, is it cheaper to rent or purchase the machine?

11) Solve for x: $3y^2x + z^2 = 0$

12) Solve for x: $2z^2x - z^3 = 1$

13) Solve for y: $2zy + 3z + 1 = 0$

14) Solve for y: $2zy + 4z + y = 2 - y$

15) Solve for z: $2zy + 3z + 1 = 0$

16) Solve for z: $2zy + 4z + y = 2 - y$

17) Solve for x: $2y^2x - y^2 - (1 + 3y) = x$

18) Solve for x: $x - y^2 + (2zy^2 + 2y)x = z^2x + z$

19) You have 27 coins, which are made up of nickels, dimes, and quarters, whose total value is \$3.30. You have three times as many quarters as dimes. How many nickels do you have?

20) A piece of wire 76 in. long is bent into the shape of a rectangle that is 2 in. longer than it is wide. What is the area of the rectangle?

21) One thousand dollars was invested, partly at 6% annual simple interest and partly at 4%. The total interest earned in the first year was \$52. How much was invested at 6%?

22) Find the side of a square such that when each side is increased by 6 in., the area of the square is increased by 156 sq in.

3.2 Equations of Degree 2 *(Quadratic Equations)*

The expression $ax^2 + bx + c$ with $a \neq 0$, is a polynomial of degree 2, and the equation $ax^2 + bx + c = 0$ is called an **equation of degree 2.** It is also called a **quadratic equation** because "quadratum" is Latin for "square." Notice that x occurs in two places, so you can't use the method of "peeling the onion." However, by using the method of completing the square, you can change the form of the equation in such a way that x occurs in only one place, and then peel away as usual. If you do that (see Appendix F), you will find that

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

This is the **quadratic formula.** It gives you the solution of that equation by simply substituting in the values of a, b, and c. No fuss, no bother, no completing the squares, no messy onion peels. It is one of the most useful formulas you will meet. **Memorize it now!** If you don't, you will stray from the straight and narrow, and not even Superman will be able to save you.

Remark The "$\pm$" sign in the quadratic formula means that there are possibly two solutions: one if we pick the "+" sign and the other if we pick the "−" sign. Sometimes the two solutions are, in fact, equal. (When does this happen?)

EXAMPLE 1 Solve for x: $x^2 - 3x + 2 = 0$.

Solution Here $a = 1$, $b = -3$, and $c = 2$. So, according to the quadratic formula

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4 \cdot 1 \cdot 2}}{2 \cdot 1},$$

which simplifies to

$$x = \frac{3 \pm \sqrt{9 - 8}}{2} = \frac{3 \pm 1}{2}.$$

When the "+" sign is used, we get the solution

$$x = \frac{3 + 1}{2} = \frac{4}{2} = 2.$$

When the "−" sign is used, we get the solution

$$x = \frac{3 - 1}{2} = \frac{2}{2} = 1.$$

So there are two solutions: $x = 1$ and $x = 2$. Can you see how useful this formula is? ■

EXAMPLE 2 Solve for x: $2x^2 - 3x + 1 = 0$.

Solution Here $a = 2$, $b = -3$, $c = 1$, and $b^2 - 4ac = 1$. So, according to the quadratic formula,

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4 \cdot 2 \cdot 1}}{2 \cdot 2},$$

which simplifies to

$$x = \frac{3 \pm \sqrt{9 - 8}}{4} = \frac{3 \pm 1}{4}.$$

The "+" sign gives $x = \dfrac{3 + 1}{4} = \dfrac{4}{4} = 1$.

The "−" sign gives $x = \dfrac{3 - 1}{4} = \dfrac{2}{4} = \dfrac{1}{2}$.

So there are two solutions: $x = 1$ and $x = \dfrac{1}{2}$. ■

EXAMPLE 3 Solve for s: $s^2 + 4s + 4 = 0$.

Solution Don't let the s fool you; this is just a quadratic equation in s, and s takes the place of x in the preceding discussion. Here $a = 1$, $b = 4$, and $c = 4$. So

$$s = \frac{-(4) \pm \sqrt{(4)^2 - 4 \cdot 1 \cdot 4}}{2 \cdot 1}$$

$$= \frac{-4 \pm \sqrt{16 - 16}}{2} = \frac{-4 \pm 0}{2} = -2.$$

In this case, there is only one root. When this happens, the root is called **repeated**, or a **root of order 2.** (When speaking of equations, **root** is another word for **solution.**) It means that the original quadratic could be factored as $s^2 + 4s + 4 = (s + 2)^2$, and yes, you guessed it: Repeated roots occur when the quadratic is a perfect square. ■

Sometimes you can solve a quadratic equation more quickly by "peeling the onion" than by using the quadratic formula. This happens when the coefficient $b = 0$. Consider the next example.

EXAMPLE 4 Solve for x: $x^2 - 9 = 0$.

Solution Here $a = 1$, $b = 0$, and $c = -9$. Instead of using the quadratic formula, we will "peel the onion." First we add 9 to both sides of the equation to get

$$x^2 = 9.$$

Now we take square roots of both sides of the equation, remembering that all positive numbers have two square roots. So

$$\pm x = \pm 3,$$

which means $x = 3$ or $x = -3$. ■

Sometimes a quadratic equation can't be solved in terms of real numbers. Consider the following.

EXAMPLE 5 Solve for y: $y^2 + 2y + 2 = 0$.

Solution Here $a = 1$, $b = 2$, and $c = 2$, so that

$$y = \frac{-2 \pm \sqrt{4 - 8}}{2} = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2\sqrt{-1}}{2} = -1 \pm \sqrt{-1}.$$

The term $\sqrt{-1}$ is not defined as a real number because no real number when squared gives a negative number. So there are no real solutions. ■

If $b^2 - 4ac < 0$, as in the last example, there are no real solutions, meaning that there are no solutions that are real numbers. But wishful thinking is a great motivator, and in this spirit, mathematicians several centuries ago agreed to consider $\sqrt{-1}$ an "imaginary number." We represent $\sqrt{-1}$ by the symbol i (for imaginary). In this case the roots of the last equation could be written as $-1 \pm i$. Any number of the form $a + ib$ is called a **complex number**, and all four basic operations (addition, subtraction, multiplication, and division) carry over from real numbers, provided we remember $i^2 = -1$. (In the 1800s, complex numbers were removed from the realm of hocus-pocus and put onto a logically sound foundation—which means that today we can all sleep much easier.) Now check the numbers $-1 \pm i$ to see whether they are in fact solutions to the problem in Example 5.

EXAMPLE 6 Verify that $y = -1 \pm i$ are solutions to $y^2 + 2y + 2 = 0$.

Solution We will check each value, one at a time.

a) If $y = -1 + i$, then

$$\begin{aligned} y^2 = (-1 + i)^2 &= (-1)^2 - 2i + i^2 \\ &= 1 - 2i - 1 \\ &= -2i \end{aligned}$$

$$2y = 2(-1 + i) = -2 + 2i.$$

Hence $y^2 + 2y + 2 = -2i - 2 + 2i + 2 = 0$. So it works!

b) If $y = -1 - i$, then

$$\begin{aligned} y^2 = (-1 - i)^2 &= (-1)^2 + 2i + (-i)^2 \\ &= 1 + 2i - 1 \\ &= +2i \end{aligned}$$

$$2y = 2(-1 - i) = -2 - 2i.$$

Hence $y^2 + 2y + 2 = 2i - 2 - 2i + 2 = 0$. ■

Remark These examples demonstrate the three possible outcomes when solving quadratic equations. The number $b^2 - 4ac$ is called the **discriminant**, since it allows you to "discriminate" among the following three cases that indicate what types of roots you will have:

a) $b^2 - 4ac > 0$: In this case, the quadratic equation has two real and distinct roots—that is, two roots that are different numbers. This was the situation in Example 1.

b) $b^2 - 4ac = 0$: Here, the square root in the solution disappears, leaving one real repeated root. This happened in Example 3.

c) $b^2 - 4ac < 0$: In this last case, the roots are complex numbers, as was the case in Example 5.

It is possible that in your travels you will meet up with a quadratic equation in x where the coefficients a, b, and c are functions of some other variables. Take the following example.

EXAMPLE 7 Solve for x: $zx + 2yx^2 + yx + z^2 - y^2 = 0$.

Solution On first appearance, this seems formidable, but by writing the x^2-term first, then the x-term, and then the term without x, we get

$$(2y)x^2 + (z + y)x + (z^2 - y^2) = 0.$$

We can see that this is a quadratic equation in x with $a = 2y$, $b = z + y$, and $c = z^2 - y^2$. Hence substituting these values into the quadratic formula gives

$$x = \frac{-(z + y) \pm \sqrt{(z + y)^2 - 8y(z^2 - y^2)}}{4y}. \quad \blacksquare$$

3.2 Exercises

Find the solutions, both real and complex, for Exercises 1–18.

1) $x^2 + 5x + 4 = 0$

2) $g^2 - 2g - 8 = 0$

3) $x^2 + 6x + 9 = 0$

4) $2y^2 + y - 1 = 0$

5) $y^2 - 8y + 16 = 0$

6) $\frac{x^2}{4} + 2x + 1 = 0$

7) $y^2 - 2y + 2 = 0$

8) $3y^2 - 4 = 0$

9) $x^2 + 3x = 4$

10) $x^2 + 7x + 4 = 0$

11) $g^2 - 4g - 5 = 0$

12) $8x^2 + 6x + 1 = 0$

13) $2y^2 + 7y + 4 = 0$

14) $y^2 - \frac{2y}{15} - \frac{1}{15} = 0$

15) $\frac{x^2}{3} + 2x - 1 = 0$

16) $y^2 + 3 = 0$

17) $x^2 + 2x = 4$

18) $3y^2 - 1.8y - 1.2 = 0$

19) Solve $\frac{1}{2}x^2 + yx - y^2 = 0$ for x.

20) Solve $\frac{x^2}{2} - 2y^2x + y^4 = 0$ for x.

21) Solve $\frac{1}{4}x^2 - (y + z)x + z^2 + y^2 = 0$ for x.

22) Solve $\frac{1}{4}x^2 - (y + z)x + z^2 + y^2 = 0$ for z.

23) A football field is 100 yd long from goal line to goal line and 160 ft wide. If a player runs diagonally across the field from one goal line to the other goal line, how far does he or she run?

24) The width of a rectangular gate is 2 ft more than its height. If a diagonal board 8 ft long is used for bracing, then what are the dimensions of the gate?

25) The path of a projectile is given by the equation $y = 2x - 0.03x^2$, where x is the horizontal distance and y is the vertical distance from the initial point, in feet. a) Find x when $y = 32$ ft. b) Physically speaking, what does it mean to have two solutions?

26) The volume of a cube is increased by 61 cu in. by increasing each edge of the cube by 1 in. Find the edge of the original cube.

27) A rectangle $6'' \times 10''$ is to be reduced in size by cutting two strips of equal width, one from the length, the other from the width, so that the area is reduced by 15 sq in. How wide should the strips be?



3.3 Solving Other Types of Equations

Some of the methods that we have used to solve equations of degree 1 and 2 can also be used to solve other kinds of equations. There are four basic tools in our equation solving toolbox:

1) *Peeling the Onion*, for any equation where the variable that we wish to solve for occurs in only one place, or any equation that can be put into that form.

2) *The Quadratic Formula*, for equations that are of "quadratic type," meaning that a well-chosen substitution can change the given equation into an ordinary quadratic equation.

3) *The Zero-Factor Property (ZFP)*, which allows us to solve equations of the form $f(x) = 0$ by factoring $f(x)$ and setting each factor equal to zero.

4) *Graphing Calculator Techniques*, which can be used to solve the equation $f(x) = 0$ by graphing $f(x)$ and seeing where the graph crosses the x-axis. Those places are the solutions of the equation. (Consult the manual that came with your calculator to see how to graph functions effectively.)

Before we start, we must emphasize the need to check the calculated solutions to make sure that they truly are solutions by substituting them into the original equation. Sometimes when manipulating an equation to find a solution, we run the risk that the "solutions" we calculated do not satisfy the original equation. Such "solutions" are called **extraneous**. Here are some examples of when extraneous solutions can creep into a problem.

1) Whenever we square both sides of an equation. For example, if we square both sides of the equation $x = 2$, we get $x^2 = 4$. The second equation has a solution $x = -2$, which clearly is not a solution to the original equation.

2) When we have denominators, which of course can't be equal to zero. Consider $\frac{x-1}{x} = \frac{-1}{x}$. Multiplying both sides of this equation by x gives the equation $x - 1 = -1$, which has the solution $x = 0$. However, this solution does not satisfy the original equation.

3) If the given equation has terms in it like $\sqrt{x+1}$, $\log(x^2 - 1)$, or $(x-2)^{\frac{3}{2}}$, we must make sure that our "solution" is a number for which such terms are defined.

4) In word problems, we must make sure that calculated solutions are physically reasonable. For example, if a word problem leads to the result that the length of a rectangle is -3 or $+5$, we must ignore the solution -3, since it does not make sense to have a negative length.

Let's begin by examining some problems of the onion-peeling variety.

EXAMPLE 1 Solve for x: $\sqrt{x} - 3 = 0$.

Solution This is not of degree 1, but it is of "onion type" because x is in one place only. To get at it, we remove the outermost layer by adding 3 to both sides to get:

$$\sqrt{x} = 3.$$

Next, we remove the square root by squaring both sides to get

$$(\sqrt{x})^2 = 3^2$$

or

$$x = 9.$$

(Don't forget to check your solution!) ■

EXAMPLE 2 Solve for x: $(\sqrt{x} + 2)^3 - 64 = 0.$

Solution As in the last example, this equation is not degree 1 but it is of "onion type" because x is in one place only. To get at it, remove the outermost layer by adding 64 to both sides:

$$(\sqrt{x} + 2)^3 = 64.$$

Next, remove the cube by taking the cube root of both sides of the equation,

$$((\sqrt{x} + 2)^3)^{1/3} = 64^{1/3}$$

or

$$\sqrt{x} + 2 = 4.$$

Subtract 2 from both sides to give

$$\sqrt{x} = 2,$$

and then square both sides to get

$$x = 4.$$

Again, remember to check your solution! ■

EXAMPLE 3 Solve for x: $\dfrac{1}{x-5} + \dfrac{1}{x+5} = \dfrac{10}{x^2 - 25}.$

Solution This is not of the onion type, but we can change that by multiplying **both sides of the equation** by $x^2 - 25$ to get rid of the denominators. Doing so, we get

$$\frac{x^2 - 25}{x - 5} + \frac{x^2 - 25}{x + 5} = 10.$$

By factoring and canceling, we get

$$x + 5 + x - 5 = 10$$

or

$$2x = 10$$

and

$$x = 5.$$

But $x = 5$ cannot be used in the original equation without dividing by zero. Hence, there is no solution to this problem. Good thing we checked. ■

Sometimes an equation can be reduced to a quadratic equation with a simple substitution. If an equation in x has three terms, one constant, one like x^m, and the other like x^{2m}, then the substitution $y = x^m$ leads to a quadratic equation in y. Consider the following example.

EXAMPLE 4 Solve for x: $x^4 - 5x^2 + 4 = 0$.

Solution This is of the quadratic type because if we use the substitution $y = x^2$, then this equation becomes

$$y^2 - 5y + 4 = 0,$$

which is quadratic and has roots $y = 1$ and $y = 4$. (Check this!) But 1 and 4 are NOT the solutions of the given equation. Instead, we have $x^2 = 1$ or $x^2 = 4$, giving us the solutions $x = \pm 1$ and $x = \pm 2$. ■

We have to be very careful, however, since taking square roots may give us solutions that are not real numbers. Consider the following example.

EXAMPLE 5 Solve for x: $x^4 - 5x^2 - 36 = 0$.

Solution Again, letting $y = x^2$, we will obtain the quadratic equation

$$y^2 - 5y - 36 = 0,$$

which has roots $y = -4$ and $y = 9$. Since $y = x^2$, we have $x^2 = -4$ and $x^2 = 9$, giving $x = \pm 2i$ and $x = \pm 3$.

However, if only real solutions are asked for, the solutions are just 3 and -3. ■

EXAMPLE 6 Solve for real x: $x^6 + 6x^3 - 16 = 0$.

Solution Now we use the substitution $y = x^3$ to obtain:

$$y^2 + 6y - 16 = 0,$$

which has solutions $y = -8$ and $y = 2$.

So: $x^3 = -8$

and $x = -2$

or: $x^3 = 2$

and $x = \sqrt[3]{2} = 2^{1/3}$. ■

EXAMPLE 7 Solve for real x: $x + \sqrt{x} - 6 = 0$.

Solution This is of the quadratic type, since if we let $w = \sqrt{x}$ we get $w^2 = x$, and so

$$w^2 + w - 6 = 0.$$

The solutions of this quadratic are $w = 2$ or -3. (Do you agree?) Therefore $\sqrt{x} = 2$ or -3, where the -3 is obviously an extraneous solution. Hence $\sqrt{x} = 2$ or $x = 4$. Check it! ■

For more complicated equations with coefficients that are also variables, the method is the same, but you should always verify (that is, check) the solutions.

The Zero-Factor Property (ZFP)

If a and b are real numbers, and $a \cdot b = 0$, then either $a = 0$, or $b = 0$, or both. (This result is used to solve equations where there are products of factors.) Consider the following.

EXAMPLE 8 Solve for x: $(x - 1)(x + 2) = 0$.

Solution Here we can use the Zero-Factor Property. By associating $x - 1$ with a and $x + 2$ with b we see that this equation is equivalent to $a \cdot b = 0$. Applying the ZFP we deduce that either $a = 0$ or $b = 0$ (or both), and so

$$x - 1 = 0$$

or

$$x + 2 = 0.$$

The solution to the first equation is clearly $x = 1$, and the solution to the second equation is $x = -2$. Hence, the solutions to the original equation are $x = 1$ or $x = -2$. (Check this!) ■

Notice that if we multiply the factors in the original equation in this example we get the quadratic equation $x^2 + x - 2 = 0$. Using the quadratic formula from the last section, we of course obtain the same solutions $x = 1$ or $x = -2$.

EXAMPLE 9 Solve for x: $(x^2 - 4)(x^2 + 2x - 3) = 0$.

Solution Applying the ZFP gives the two equations

$$x^2 - 4 = 0$$

or

$$x^2 + 2x - 3 = 0.$$

Each of these equations is quadratic and can be solved using the quadratic formula. The solutions to the first equation are $x = 2$ or $x = -2$. (Make sure you verify this for yourself!)

The solutions to the second equation are $x = -3$ or $x = 1$. (Check this also!) So the solutions to the original equation are $x = 2, x = -2, x = -3$, or $x = 1$. ■

The ZFP can be extended to cases when there are more than two factors. For example, if $a \cdot b \cdot c = 0$, then either $a = 0$, $b = 0$, or $c = 0$. When you have a polynomial equation $f(x) = 0$, the idea is to factor $f(x)$ as far as possible and then to apply the ZFP. Remember that the right-hand side **must be zero** for the ZFP to apply. If your factoring skills are a little rusty, you can buff them up in Chapter 5.

EXAMPLE 10 Solve for x: $x^5 + 3x^4 - 4x^3 = 0$.

Solution We can factor this by first taking out a common factor, to get

$$x^3(x^2 + 3x - 4) = 0.$$

Next, the second factor can be factored further to give

$$x^3(x + 4)(x - 1) = 0.$$

The ZFP says that

$$x^3 = 0$$
$$x + 4 = 0$$

or

$$x - 1 = 0.$$

Hence $x = 0, -4$, or 1. Remember to check the solutions. Again, factoring methods are given in Chapter 5. ■

3.3 Exercises

Find the solutions, both real and complex, for Exercises 1–21. Check your answers.

1) $(x - 3)(x + 6) = 0$

2) $(x + 2)(x - 9) = 0$

3) $(2x + 1)(x - 2) = 0$

4) $(2x + 3)(3x - 2) = 0$

5) $(x^2 - 9)(x + 6) = 0$

6) $(x - 1)(x + 2)(x - 3) = 0$

7) $(x^2 - 5)(x^2 - 16) = 0$

8) $(x^2 - 1)(x^2 + 1) = 0$

9) $(x^2 + 1)\left(\frac{x}{3} - 4\right) = 0$

10) $(x^2 + 4x - 5)(3x^2 - 81) = 0$

11) $z^4 - 8z^2 + 16 = 0$

12) $s^4 - 5s^2 + 4 = 0$

13) $x^4 - 64 = 0$

14) $(x^2 - 4x - 5)(3x^2 + 81) = 0$

15) $s^4 - 9s^2 + 20 = 0$

16) $\sqrt{x} - 4 = 0$

17) $\sqrt{x} - 3 = 5 - \sqrt{x}$

18) $(\sqrt{2x} - 3)^3 + 1 = 0$

19) $\left(\frac{1}{\sqrt{x}} - 3\right)^4 - 1 = 0$

20) $\frac{1}{\sqrt[3]{x^4}} = \frac{1}{\sqrt[3]{x}}$

21) $\frac{4}{\sqrt{x^3}} = \frac{1}{\sqrt{x}}$

22) Find all real solutions to $x^{2/5} - 3x^{1/5} + 2 = 0$.

23) Find all solutions to $\frac{1}{x + 1} + \frac{1}{x} = \frac{3}{x^2 + x}$.

(*Hint:* $x^2 + x$ is a common denominator for the two algebraic fractions on the LHS.)