1.1 Exercises

Simplify:

1) $4 - (5 - 3) + 3(4 - 7) - 4(1 + 2)$

2) $5 + (3 - 5) - (4 - 2) - (-5 - 3)$

3) $-2(3 - 4) + 4(5 + 3) - 3(2 - 6)$

4) $2 - (3 - \frac{1}{4}) + (2.4 - 3.2) - (1.2 - 2.3)$

5) $4xy - (x - 2xy) - 2y(x - 1)$

6) $(s - t) - (u - t) - (v - u) - (s - v)$

7) $2x(y - 3) - y(x + xy) + 2y(x + 1)$

8) $x(y + z) - z(x + y) + 2y(x - z) - x(3y - 2z)$

9) $xy(x + y^2) - (2x^2y^2 - 2xy^3) - 2y^2(x^2 - y + xy)$

10) $xy^2(x^2 + y^2) - 3x(2xy^2 - 2xy^3) + 2y^2(x - xy^2 - x^2)$

11) Consider simplifying the expression $-2(7 - 4) + 5(2 + 3) - 2(2 - 6)$ by a) first computing the expression within the parentheses and b) then using the distributive law. Keep track of the total number of computations (additions, subtractions, or multiplications) in each method. Which way is "cheaper"?



1.2 Multiplying and Dividing Fractions

(We'll do adding and subtracting later.)

Multiplying fractions is the easiest manipulative task. The numerator is the product of the given numerators, and the denominator is the product of the given denominators. That is,

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}.$$

We use the symbol " $\cdot$ " to denote multiplication, although sometimes we will omit the symbol altogether.

EXAMPLE 1 a) $\frac{2}{3} \cdot \frac{5}{7} = \frac{2 \cdot 5}{3 \cdot 7} = \frac{10}{21}$

b) $\frac{1}{9} \cdot \left(-\frac{5}{8}\right) = \frac{1}{9} \cdot \frac{-5}{8} = \frac{1 \cdot (-5)}{9 \cdot 8} = \frac{-5}{72}$ ■

This rule can be extended to multiplying more than two fractions simply by multiplying across all the numerators and denominators.

EXAMPLE 2 a) $\frac{1}{4} \cdot \frac{7}{5} \cdot \frac{3}{8} = \frac{1 \cdot 7 \cdot 3}{4 \cdot 5 \cdot 8} = \frac{21}{160}$

b) $\frac{-1}{7} \cdot \frac{3}{8} \cdot \frac{-2}{\pi} = \frac{(-1)(3)(-2)}{7 \cdot 8 \cdot \pi} = \frac{6}{56\pi} = \frac{3}{28\pi}$ ■

Multiplying a number by 1 produces the same number, so, for example: $\frac{5}{6} \cdot 1 = \frac{5}{6} \cdot \frac{4}{4} = \frac{20}{24}$. Going backward is usually more important: $\frac{20}{24} = \frac{5 \cdot 4}{6 \cdot 4} = \frac{5 \cdot \cancel{4}}{6 \cdot \cancel{4}} = \frac{5}{6}$. Here's the key point: You can cancel the factor 4 in the numerator and the denominator. More generally, if a number $c \neq 0$ is a factor of both the top and bottom of a fraction, it may be canceled. When all those common factors are canceled, the fraction is said to be in **lowest terms.**

EXAMPLE 3 Put $\frac{30}{84}$ into lowest terms.

Solution First factor the numerator and denominator as much as possible and then cancel all common factors.

$$\frac{30}{84} = \frac{\cancel{2} \cdot \cancel{3} \cdot 5}{2 \cdot \cancel{2} \cdot \cancel{3} \cdot 7} = \frac{5}{2 \cdot 7} = \frac{5}{14} \quad \blacksquare$$

WARNING: Make sure you cancel only those numbers that are factors of the entire top and the entire bottom. Don't be tempted to try "creative canceling." Consider the expression

$$\frac{3 + x^2}{3}.$$

Can you cancel the 3s? **NO NO NO NO NO NO NO NO NO!!** Get the picture? The problem is that 3 is not a factor of the **entire** numerator, only of its first term. Hence you cannot simply cancel the 3s. But, if the expression had been, for example,

$$\frac{3 + 3x^2}{3},$$

then you could write:

$$\frac{3 + 3x^2}{3} = \frac{3(1 + x^2)}{3},$$

and now you could cancel the 3s to get $1 + x^2$.

Dividing by a fraction is done by inverting **that** fraction and multiplying:

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \cdot \frac{d}{c} = \frac{a \cdot d}{b \cdot c}.$$

For example,

$$\frac{\frac{-1}{3}}{\frac{5}{6}} = \frac{-1}{3} \cdot \frac{6}{5} = \frac{-6}{15} = \frac{-2}{5},$$

or you can cancel early:

$$\frac{-1}{\cancel{3}_1} \cdot \frac{\cancel{6}^2}{5} = \frac{-2}{5}.$$

EXAMPLE 4 Simplify the following expressions:

a) $\dfrac{\frac{2}{3}}{\frac{3}{8}}$ b) $\dfrac{\frac{5}{4}}{\frac{-10}{3}}$

Solution a) $\dfrac{\frac{2}{3}}{\frac{3}{8}} = \frac{2}{3} \cdot \frac{8}{3} = \frac{16}{9}$ b) $\dfrac{\frac{5}{4}}{\frac{-10}{3}} = \frac{1\cancel{5}}{4} \cdot \frac{3}{-\cancel{10}_2} = -\frac{3}{8}$ ■

EXAMPLE 5 Simplify the expression $\left(\dfrac{x+1}{y}\right) \cdot \left(\dfrac{-y+1}{x}\right)$.

Solution $\left(\dfrac{x+1}{y}\right) \cdot \left(\dfrac{-y+1}{x}\right) = \dfrac{(x+1) \cdot (-y+1)}{y \cdot x}$

$$= \frac{-xy + x - y + 1}{xy}$$

$$= \frac{1 - xy + x - y}{xy}$$ ■

EXAMPLE 6 Simplify the expression $\dfrac{\frac{x^2y}{z}}{\frac{xy^2}{z^3}}$.

Solution $\dfrac{\frac{x^2y}{z}}{\frac{xy^2}{z^3}} = \dfrac{x^2y}{z} \cdot \dfrac{z^3}{xy^2}$

$$= \frac{x^2yz^3}{zxy^2} = \frac{xz^2}{y}$$ ■

Again, if you are confused by the exponents, skip ahead to Section 1.4 and then return here.

In calculus, we use numbers called ***real numbers***. These can be thought of as all decimal numbers, including those having an infinite number of digits. The set of real numbers corresponds to the set of points on the number line. Rational numbers are those real numbers that can be expressed as a quotient of two integers. (Integers are the numbers 0, 1, −1, 2, −2, etc.) For example, the numbers $\frac{1}{3}$, $\frac{4}{10}$, -5, $\frac{18}{7}$, $\sqrt{64}$, and 3.857 are all rational numbers. (Convince yourself of this by expressing them as the quotient of two integers.) All other real numbers are called irrational. The number $\sqrt{2}$ can be easily shown to be irrational. The number π is also irrational, but that's much harder to prove.

1.2 Exercises

In Exercises 1–14, simplify and reduce to lowest terms. (Some answers may have more than one form.)

1) $\dfrac{5}{16} \cdot \dfrac{8}{10}$

2) $\dfrac{2\pi}{3} \cdot \dfrac{3\pi}{4}$

3) $\dfrac{-1}{3} \cdot \dfrac{-9}{5}$

4) $\dfrac{15}{28} \cdot \dfrac{12}{5} \cdot \dfrac{7}{8}$

5) $\dfrac{\dfrac{4}{75}}{\dfrac{8}{25}}$

6) $\dfrac{\dfrac{-7}{51}}{\dfrac{3}{12}}$

7) $\dfrac{\dfrac{3\pi}{7}}{\dfrac{2\pi}{3}}$

8) $\dfrac{x}{y} \cdot \dfrac{s}{u} \cdot \dfrac{y}{s} \cdot \dfrac{u}{x}$

9) $\dfrac{7x}{3y} \cdot \dfrac{3y+2}{x}$

10) $\left(\dfrac{x+2}{1+y}\right) \cdot \left(\dfrac{1-y}{x}\right)$

11) $\dfrac{\dfrac{xy}{w}}{\dfrac{xy-2x}{w}}$

12) $\dfrac{xy}{wz} \cdot \dfrac{w^2z}{x^2y^2}$

13) $\dfrac{\dfrac{xy}{(x+y)}}{\dfrac{x^2y}{(x+y)^3}}$

14) $\dfrac{\dfrac{xy}{(x-y)}}{\dfrac{x^2}{y} \cdot \dfrac{y^3}{x}}$

15) Draw the number line, and locate the following numbers:

a) $\dfrac{-126}{106}$ b) $-3+\dfrac{10}{27}$ c) $\dfrac{\pi}{2}-1$

16) Express the numbers -5, $\sqrt{64}$, and 3.857 as a quotient of two whole numbers.



Adding and Subtracting Fractions

Adding and subtracting fractions is easy when the denominators are all the same. For example,

$$\frac{2}{5}+\frac{17}{5}-\frac{4}{5}=\frac{2+17-4}{5}=\frac{15}{5}=3.$$

But what if the fractions don't all have the same denominator? Then you first need to rewrite the fractions so that they all have the same denominator, called a **common denominator**. For example, if you want to add $\frac{2}{3}+\frac{4}{5}$, you can use a common denominator of 15. So

$$\frac{2}{3}+\frac{4}{5}=\frac{10}{15}+\frac{12}{15}=\frac{22}{15}.$$

Remember, to get a common denominator you can always use the product of the individual denominators ($3 \cdot 5 = 15$), but sometimes even a smaller number will do it, as long as both given denominators divide evenly into it. Then, to get the new numerator,

$$\frac{2}{3}=\frac{?}{15},$$

divide 3 into 15, to get 5, then multiply by 2, to get 10. Try it: $\frac{3}{7}=\frac{?}{42}$. (We're doing the opposite of canceling common factors.)

EXAMPLE 1 a) Simplify $\dfrac{3}{5}+\dfrac{1}{2}-\dfrac{2}{3}$.

Solution $5 \cdot 2 \cdot 3 = 30$ will serve as a common denominator. So

$$\frac{3}{5}+\frac{1}{2}-\frac{2}{3}=\frac{18}{30}+\frac{15}{30}-\frac{20}{30}$$

$$=\frac{18+15-20}{30}=\frac{13}{30}$$

b) Simplify $\frac{1}{6} - \frac{1}{9}$.

Solution We could use $6 \cdot 9 = 54$ as a common denominator, but even 18 will do nicely because **both 6 and 9 divide evenly into 18.** So

$$\frac{1}{6} - \frac{1}{9} = \frac{3}{18} - \frac{2}{18} = \frac{1}{18}.$$

c) Simplify $\dfrac{\frac{1}{2} + \frac{3}{4}}{\frac{1}{3} - \frac{1}{6}}$.

Solution $\dfrac{\frac{1}{2} + \frac{3}{4}}{\frac{1}{3} - \frac{1}{6}} = \dfrac{\frac{2+3}{4}}{\frac{2-1}{6}} = \dfrac{\frac{5}{4}}{\frac{1}{6}} = \dfrac{5}{4} \cdot \dfrac{6}{1} = \dfrac{15}{2}$ ■

You will also have to be comfortable with adding and subtracting fractional expressions involving variables.

EXAMPLE 2 $\dfrac{xy}{z} + \dfrac{x}{5} = \dfrac{5xy}{5z} + \dfrac{xz}{5z} = \dfrac{5xy + xz}{5z}$ ■

EXAMPLE 3

$$\frac{\frac{3a+2b}{5ab}}{\frac{a}{b} - \frac{b}{a}} = \frac{\frac{3a+2b}{5ab}}{\frac{a^2 - b^2}{ab}}$$

$$= \frac{3a+2b}{5ab} \cdot \frac{ab}{a^2 - b^2}$$

$$= \frac{3a+2b}{5a^2 - 5b^2}$$ ■

1.3 Exercises

Express as a single fraction and simplify:

1) $\frac{1}{3}+\frac{1}{4}$

2) $\frac{7}{6}+\frac{5}{24}$

3) $\frac{2}{5}-\frac{1}{2}+\frac{1}{3}$

4) $\frac{1}{2}-\frac{1}{4}+\frac{1}{8}-\frac{1}{16}$

5) $\frac{1}{2}+\frac{4}{3}-\frac{2}{5}-\frac{3}{15}$

6) $\frac{5}{6}-\frac{4}{3}+\frac{2}{9}-\frac{3}{2}$

7) $\dfrac{\frac{1}{3}+\frac{2}{5}}{\frac{3}{2}}$

8) $\dfrac{\frac{1}{4}-\frac{2}{3}}{\frac{3}{2}-\frac{2}{5}}$

9) $\dfrac{\frac{2}{7}+\frac{1}{3}}{\frac{4}{3}+\frac{2}{5}}+\frac{1}{3}$

10) $\frac{2}{15}\left(\frac{1}{3}-\frac{2}{9}\right)-\frac{1}{3}\left(\frac{1}{6}-\frac{4}{9}\right)+\frac{1}{5}\left(\frac{2}{9}-\frac{1}{3}\right)$

11) $\frac{1}{x}+\frac{1}{y}$

12) $\frac{1}{y}-\frac{1}{x}$

13) $\frac{4}{x}-\frac{2}{y}+\frac{1}{z}$

14) $\frac{s}{tu}+\frac{t}{u}+\frac{u}{s}$

15) $\frac{1}{x}-\frac{x+1}{xy}+\frac{x-2}{xz}$

16) $\dfrac{\frac{1}{x}-\frac{1}{y}}{\frac{1}{x}+\frac{1}{y}}$

17) $\dfrac{\frac{1}{y}-\frac{x}{z}}{\frac{1}{z}-\frac{1}{x}}$

18) $\frac{1}{x}\left(\frac{1}{y}-\frac{1}{z}\right)-\frac{1}{y}\left(\frac{1}{x}-\frac{1}{z}\right)$

19) $\dfrac{\frac{1}{st}-\frac{1}{w}}{\frac{1}{tw}-\frac{2}{s}}$

20) $\frac{2}{x}+x\left(\frac{3}{xy}-\frac{4}{y}\right)-y\left(\frac{2}{x}-\frac{2}{xy}\right)$

21) $\frac{4yz}{x^2}-\frac{2z}{xy^2}+\frac{1}{xyz}$

22) $\dfrac{\frac{1}{x}-\frac{x}{y}}{\frac{2y}{x}+\frac{2x}{y}}+\frac{x-y}{xyz}$

1.4 Exponents

Positive whole number exponents are a simple mathematical notation to represent repeated multiplication by the same factor. But you can also have negative numbers and 0 as exponents. Consider this table.

Exponents increase by 1. ↑ Exponents decrease by 1. ↓

3^4	$= 3 \cdot 3 \cdot 3 \cdot 3$	$= 81$
3^3	$= 3 \cdot 3 \cdot 3$	$= 27$
3^2	$= 3 \cdot 3$	$= 9$
3^1	$= 3$	$= 3$
3^0	$= 1$	$= 1$
3^{-1}	$= \frac{1}{3}$	$= \frac{1}{3}$
3^{-2}	$= \frac{1}{3^2}$	$= \frac{1}{9}$
3^{-3}	$= \frac{1}{3^3}$	$= \frac{1}{27}$
3^{-4}	$= \frac{1}{3^4}$	$= \frac{1}{81}$

Numbers are multiplied by 3. ↑ Numbers are divided by 3. ↓

See the pattern? Let's summarize: $3^n = \underbrace{3 \cdot 3 \cdot 3 \cdot \ldots \cdot 3 \cdot 3}_{n \text{ factors}}$, $3^{-n} = \frac{1}{3^n}$, and $3^0 = 1$.

This works not only for the number 3; for any number a and any positive integer n, we have:

$$a^n = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a \cdot a}_{n \text{ factors}}$$

$$a^{-n} = \frac{1}{a^n} \text{ for } a \neq 0,$$

and

$$a^0 = 1 \text{ for } a \neq 0.$$

In all of these rules, the number a is called the **base**, while the number n is called the **exponent**. There are several ways to simplify complicated expressions. Here are the laws of exponents that will help you. Know what they say **and what they don't say**.

Laws of Exponents

Let m and n be any positive or negative integers, or 0. Let a and b be any real numbers. Then in each of the following, if the expressions on both sides exist, we have:

1) $a^m \cdot a^n = a^{m+n}$

2) $\dfrac{a^m}{a^n} = a^{m-n}$

3) $(a^m)^n = a^{m \cdot n}$

4) $(a \cdot b)^n = a^n \cdot b^n$

5) $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

6) $\left(\dfrac{a}{b}\right)^{-n} = \left(\dfrac{b}{a}\right)^n$

(When might some of these expressions not exist?)

These rules are also true for values of m and n that are not integers. We will deal with such exponents in the next section.

EXAMPLE 1 a) $\dfrac{4^2 - 1}{3^3 - 2^2} = \dfrac{16 - 1}{27 - 4} = \dfrac{15}{23}$

b) $5^{-1} + 3^{-1} = \dfrac{1}{5} + \dfrac{1}{3} = \dfrac{3 + 5}{15} = \dfrac{8}{15}$

c) $\dfrac{3 \cdot 8^2}{9 \cdot 8^3} = \dfrac{3}{9} \cdot \dfrac{8^2}{8^3} = \dfrac{1}{3} \cdot \dfrac{8^2}{8^3} = \dfrac{1}{3} \cdot \dfrac{1}{8} = \dfrac{1}{24}$ ■

EXAMPLE 2
$$\frac{x^2y^5}{x^{-3}} \div \frac{x^{-5}y^4}{x^3} = \left(\frac{x^2}{x^{-3}}y^5\right) \div \left(\frac{x^{-5}}{x^3}y^4\right)$$
$$= x^5y^5 \div x^{-8}y^4$$
$$= \frac{x^5y^5}{x^{-8}y^4} = x^{13}y \quad \blacksquare$$

EXAMPLE 3
$$\frac{1}{a^3} - \left(\frac{1}{a^5} - \frac{1}{a^2}\right) = \frac{1}{a^3} - \frac{1}{a^5} + \frac{1}{a^2}$$
$$= \frac{a^2 - 1 + a^3}{a^5} \quad \blacksquare$$

EXAMPLE 4 $\left(\dfrac{x^{-2}}{x^8}\right)^{-2} = (x^{-2}x^{-8})^{-2}$

$= (x^{-10})^{-2} = x^{20}$ ■

EXAMPLE 5 a) $(x^2y^3)^5 = (x^2)^5(y^3)^5 = x^{10}y^{15}$

b) $\left(\dfrac{a^{-3}}{b}\right)^4 = \dfrac{(a^{-3})^4}{b^4} = \dfrac{a^{-12}}{b^4} = \dfrac{1}{a^{12}b^4}$

c) $\left(\dfrac{x^5}{y^3}\right)^{-2} = \left(\dfrac{y^3}{x^5}\right)^2 = \dfrac{(y^3)^2}{(x^5)^2} = \dfrac{y^6}{x^{10}}$ ■

EXAMPLE 6 Can we simplify $(x^2 + y^3)^7$?

Solution The laws of exponents are about quantities that are multiplied, divided, or exponentiated but **not added or subtracted**. So there is no way of writing the above expression in a way that would be called simplified. (However, check out the Binomial Theorem in Appendix E for a different way of writing this expression.)

EXAMPLE 7 a) $(x^2y^2)^{10} = x^{20}y^{20}$ (Easy!)

b) $(x^2 + y^2)^{10}$ can't be changed using exponent laws. See Example 6.

In any event, it is NOT equal to $x^{20} + y^{20}$. ■

1.4 Exercises

In Exercises 1–9, simplify the expressions.

1) $\dfrac{4^{-1}5^2}{2^23^{-2}}$

2) $\dfrac{3^52^3}{4^23^3}$

3) $\dfrac{5^3}{3^{-1}5^2 + 4^{-1}5^3}$

4) $\dfrac{2^3}{4^{-1}2^2 + 3^{-1}2^3}$

5) $4^3\left(\dfrac{1}{4}\right)^2 3^{-4}$

6) $\left(\dfrac{x^{-4}}{x^{-7}}\right)^{-2}$

7) $\dfrac{1}{2^{-3}} - \dfrac{1}{2} + \dfrac{1}{5^{-2}}$

8) $\dfrac{1}{3^{-2}} - \dfrac{1}{3} + \dfrac{1}{4^{-1}}$

9) $x^2y^{-2}z^3x^{-2}y^3z^5$

10) Show by example that $(x^{-2} + y^{-2})^2 \neq x^{-4} + y^{-4}$; that is, find values for x and y so that the two sides are unequal for those values. (*Hint:* Just dive in and try some. Maybe you'll be lucky.)

Simplify using only positive exponents:

11) $\dfrac{x^{-1}y^2}{y^2x^{-2}}$

12) $\dfrac{(x^3y^{-2})^6}{(y^{-5}x^{-2})^{-3}}$

13) $\dfrac{(x^2y^{-3})^2}{(y^{-3}x^{-2})^{-2}}$

14) $\dfrac{x^4y^2}{x^{-3}} \div \dfrac{x^3y^{-2}}{y^5}$

15) $\dfrac{x^2y}{x^3} \div \dfrac{x^{-3}y^6}{y^4}$

16) $\dfrac{\dfrac{1}{x^2} - \dfrac{1}{y^3}}{\dfrac{1}{x^3} + \dfrac{1}{y^2}}$

17) $\dfrac{\dfrac{x^2y^{-3}}{3z^2} - \dfrac{z^{-3}y^{-3}}{3x^2}}{\dfrac{x^{-4}y^2}{3z^{-2}}}$

18) $\dfrac{2}{x^2} + x\left(\dfrac{3}{x^2y^2} - \dfrac{4x}{y}\right) - y\left(\dfrac{2y}{x^2} - \dfrac{2}{xy^2}\right)$

19) $(x^{-1} + y^{-1})^{-1}$

20) $\dfrac{\dfrac{x^3y^{-2}}{2z^5} - \dfrac{z^{-2}x^{-3}}{y^4}}{\dfrac{x^2y^{-2}}{2z^2}}$

21) $\left(\dfrac{x^{-2}}{x^{-3}}\right)^{-4}$

22) The body-mass index, abbreviated BMI, is given by the formula $BMI = 703wh^{-2}$, where w is the person's weight in pounds and h is the person's height in inches. Find the BMI for Charles Barkley, who played in the 1992 Olympics at 6 ft 6 in. and 250 pounds.



Roots *(Also Called Radicals)*

DEFINITION

We say that the number y is a **square root** of the number x if $y^2 = x$. A **cube root** of the number x is a number y such that $y^3 = x$, and so on. In general, if n is any positive integer, we say that the number y is an "nth root" of the number x if $y^n = x$. The number n is called the **order** of the root.

For Roots of Even Order

1) The number 16 has two square roots, 4 and -4. The "radical" symbol $\sqrt{\ }$ always means the **positive** square root! So, $\sqrt{16} = 4$, but $\sqrt{16} \neq -4$. We say that 16 has two square roots: $\sqrt{16}$ and $-\sqrt{16}$ (i.e., 4 and -4). The situation is similar for higher even-order roots: Each positive number x has two even-order nth roots, denoted $\pm\sqrt[n]{x}$, where $\sqrt[n]{x}$ is always taken as the **positive** nth root.

For example: $\sqrt[4]{16} = 2$, since $2^4 = 16$; also $\sqrt[6]{64} = 2$, $\sqrt[4]{81} = 3$, and $-\sqrt{81} = -9$.

2) Since no real number multiplied by itself an even number of times can produce a negative number, negative numbers have no real roots of even order. So $\sqrt{-4}$ is not a **real** number. (Do you know what it is?) Notice that we usually write $\sqrt{\ }$ instead of $\sqrt[2]{\ }$.

To sum up:

1) Every positive number x has two real nth roots if n is even—namely, $\sqrt[n]{x}$ and $-\sqrt[n]{x}$.

2) Only positive numbers, and 0, have even-order roots that are real numbers.

For Roots of Odd Order

Things are a little different with odd-order roots. All numbers have exactly one odd-order nth root, denoted $\sqrt[n]{x}$. The number 8 has exactly one real cube root, namely, 2. So, $\sqrt[3]{8} = 2$. But notice that $\sqrt[3]{-8} = -2$, because $(-2)^3 = -8$, and $\sqrt[3]{-27} = -3$, because $(-3)^3 = -27$. Get it? Multiplying a negative number by itself an odd number of times produces a negative number!

An alternative notation for $\sqrt[n]{a}$ is $a^{1/n}$. Both notations mean exactly the same thing. For example, $8^{1/3} = 2$, $25^{1/2} = 5$ (not -5), and $(-16)^{1/2}$ is not defined (as a real number).

DEFINITION

We can also define **fractional exponents**: If $\frac{m}{n}$ is in lowest terms, and if n and a are such that $\sqrt[n]{a}$ makes sense, then we define $a^{m/n}$ to be $(\sqrt[n]{a})^m$, which equals $\sqrt[n]{a^m}$. If a is much larger than 1, it is usually easier to take the root first; it keeps the numbers down.

EXAMPLE 1

a) $8^{2/3} = (\sqrt[3]{8})^2 = 2^2 = 4$

b) $\left(\frac{-1}{27}\right)^{4/3} = \left(\sqrt[3]{\frac{-1}{27}}\right)^4 = \left(\frac{-1}{3}\right)^4 = \frac{1}{81}$

c) $(-32)^{4/5} = (\sqrt[5]{-32})^4 = (-2)^4 = 16$ ■

Laws of Exponents

Let r and s be any rational numbers. Let a and b be any real numbers. Then, in each of the following, if the expressions on both sides exist, they will be equal. (When might they not exist?)

1) $a^r \cdot a^s = a^{r+s}$

2) $\dfrac{a^r}{a^s} = a^{r-s}$

3) $(a^r)^s = a^{r \cdot s}$

4) $(ab)^r = a^r b^r$

5) $\left(\dfrac{a}{b}\right)^r = \dfrac{a^r}{b^r}$

6) $\left(\dfrac{a}{b}\right)^{-r} = \left(\dfrac{b}{a}\right)^r = \dfrac{b^r}{a^r}$

Remark (*Very important*) Look at law #4. Let $r = \frac{1}{2}$. Then we have $(ab)^{1/2} = a^{1/2}b^{1/2}$, and so $\sqrt{ab} = \sqrt{a}\sqrt{b}$. In words, the square root of a product is the product of the square roots. From law #5 we have $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$, or the square root of a quotient is the quotient of the square roots. Nowhere are there laws that say that $\sqrt{a+b}$ equals $\sqrt{a} + \sqrt{b}$, **because it doesn't**. The square root of a sum is not the sum of the square roots. Also $\sqrt{a-b}$ is not $\sqrt{a} - \sqrt{b}$.

We can generalize the above remark to nth-order roots in the following two root laws.

Root Laws

Let n be a positive integer. Let a and b be real numbers. Then, as long as both sides exist, we have:

1) $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$

2) $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$.

(That's all. There are no similar laws for addition or subtraction.)

1.5 Exercises

In Exercises 1–25, simplify the expressions as much as possible, using rational exponent notation where appropriate.

1) $\sqrt{144}$

2) $\sqrt{169}$

3) $\sqrt[3]{-64}$

4) $\sqrt{\dfrac{1}{9}}$

5) $\sqrt[5]{-32}$

6) $\sqrt[5]{-243}$

7) $\sqrt{\frac{4}{49}}$

8) $\sqrt[3]{\frac{8}{27}}$

9) $8^{5/3}$

10) $(-8)^{5/3}$

11) $(-32)^{2/5}$

12) $-(32)^{2/5}$

13) $\left(\frac{16}{9}\right)^{-3/2}$

14) $\left(\frac{25}{16}\right)^{-3/2}$

15) $(0.01)^{-3/2}$

16) $(0.008)^{4/3}$

17) $2^{5/3}2^{3/5}$

18) $8^{5/3}4^{3/2}$

19) $(3^{2/3})^{3/4}$

20) $\frac{2^{4/7}}{2^{3/2}}$

21) $3(27^{2/3})^{5/2}$

22) $\frac{(2^{1/3})^{2/5}}{\sqrt[5]{2}}$

23) $\sqrt{64x^8}$

24) $3^{1/3}9^{1/3} + 2^{1/3}16^{1/6}$

25) $\frac{x^{1/3}y^{2/5}}{\sqrt[3]{xy}}$

26) If $a > 0$ and $b > 0$, is $\sqrt{a^2b^2}$ the same as ab? Justify your answer.

27) If $a > 0$ and $b > 0$, is $\sqrt{a^2 + b^2}$ the same as $a + b$? Justify your answer.

28) If $a > 0$ and $b > 0$, is $\sqrt{a^2 - b^2}$ the same as $a - b$? Justify your answer.

29) Is $\sqrt[3]{x^3 - 8}$ the same as $x - 2$? Justify your answer.

30) If $x^2 + y^2 = 25$, can we conclude that $x + y = 5$? Why or why not?

Simplify if possible:

31) $\sqrt{25x^4}$

32) $\sqrt[3]{-27x^6}$

33) $(4x^6)^{3/2}$

34) $\left(\frac{9x^8}{16y^4}\right)^{-1/2}$

35) $(81x^2 - 4y^2)^{-1/2}$



Percent

Percent, represented by the symbol %, means "per hundred." So 5% of 400 means 5 one-hundredths of 400, which is $\left(\frac{5}{100}\right)(400) = 20$. In general, $x\%$ of y is $\left(\frac{x}{100}\right)(y)$.

EXAMPLE 1 Find 15% of 90.

Solution $\frac{15}{100} \cdot 90 = \frac{15 \cdot 9}{10} = \frac{135}{10} = 13.5$ ■

EXAMPLE 2 Find 1% of 320.

Solution $\frac{1}{100} \cdot 320 = \frac{320}{100} = 3.2$

Note that 1% of any number is always $\frac{1}{100} = .01$ of the number, so move the decimal point over two places to the left. (For example, 1% of 4567 is 45.67 and 1% of 378.2 is 3.782.) ■

EXAMPLE 3 Find $\frac{1}{3}\%$ of 930.

Solution Since 1% of 930 is 9.3, $\frac{1}{3}\%$ of 930 would be $\frac{1}{3}$ of 9.3, or 3.1.

Alternatively: $\frac{\frac{1}{3}}{100} \cdot 930 = \frac{930}{300} = \frac{93}{30} = \frac{31}{10} = 3.1$ ■

EXAMPLE 4 Find 250% of 150.

Solution $\frac{250}{100} \cdot 150 = \frac{250 \cdot 15}{10} = 25 \cdot 15 = 375$ ■

EXAMPLE 5 A CD usually sells for \$15.99. Shower Records is having a big sale, offering 60% off. How many CDs can you get for \$20, and exactly how much would you pay (assuming no sales tax)?

Solution A 60%-off sale means that the CDs will cost 40% of their original price. Now 40% of \$15.99 is $\frac{40}{100} \cdot (\$15.99) = \frac{4 \cdot (\$15.99)}{10} = \$6.396 = \$6.40$ (rounded up to the nearest cent). So you can buy three CDs at a cost of \$19.20. ■

EXAMPLE 6 In 1995, the federal budget called for a 30% cut in the deficit to $147 billion. What would the deficit have been if it had not been cut?

Solution Let x = the uncut deficit.

So 70% of x is 147 billion, that is,

$$\frac{70}{100} \cdot x = 147 \text{ billion.}$$

Solving for x:

$$x = \frac{100}{70} \cdot (147 \text{ billion})$$

$$= 210 \text{ billion.}$$

So the deficit would have been $210 billion. ■

1.6 Exercises

1) Find 25% of 200.

2) Find 3.3% of 7.1.

3) The number 587 is 45% of what number?

4) The number 58 is 22% of what number?

5) The bookstore is having an inventory sale with a 25% reduction in prices. How much will a $16.99 sweatshirt cost you? How about a $1499.00 computer?

6) A hardware store is having a sale with a 15% reduction in prices. How much will a $78 drill cost you?

7) After a 40% cost reduction, a textbook that you purchased cost $20.40. What was the original price?

8) After a 1/3-off cost reduction, your new printer cost $98. What was the original price?

9) A hardware store is having a sale with a 20% reduction in prices. How much will a $239 table saw cost you?

10) After a 65% cost reduction, your new skates cost $84. What was the original price?

11) Joe heard that the government was selling foreclosed land at 35% off. He went to look at a farm going for the reduced price of $97,500. Joe bought the farm. How much did he save?

12) Diana borrowed $100,000 to start a new business. The loan had an interest rate of "prime plus two," with only the interest to be paid once a year. (Nice if you can get it.) The prime rate in the first year was 6%, and so her interest rate was 8%.

   a) How much interest did she pay at the end of the first year?

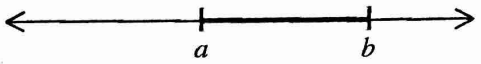
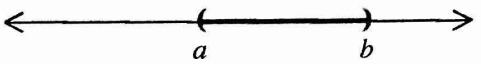
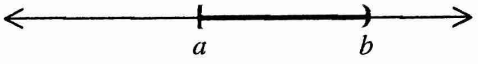
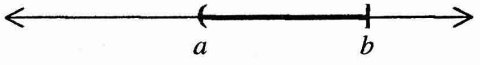
   b) The prime rate in the second year rose to 9%. How much did she pay after the second year, assuming the principal remains untouched?

   c) How much did the interest payment increase from the first year to the second?

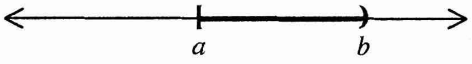
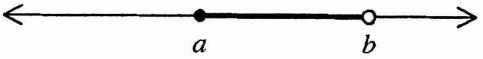
   d) What was the percentage increase?

1.8 Intervals

An **interval** is just a connected piece of the number line. For example, the set of all real numbers between 0 and 1, including 0 and 1, is called a **closed interval** and is denoted [0, 1]. If you don't mean to include 0 and 1, you have an **open interval** and express it by using round parentheses, (0, 1). If 0 is to be included, but not 1, you write [0, 1). Each of these intervals can also be shown on the number line, or expressed as a pair of inequalities. In general, if $a < b$:

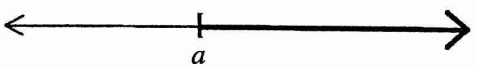
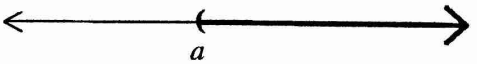
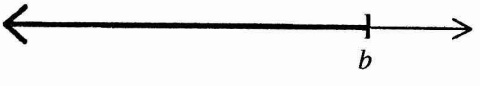
Interval Notation	Number Line	Inequalities
$[a, b]$		$a \le x \le b$
(a, b)		$a < x < b$
$[a, b)$		$a \le x < b$
$(a, b]$		$a < x \le b$

Note Some books use a different notation for an interval on the number line.

Instead of  they use .

Notice the "missing" point at b

All of these are called **finite intervals** because their length is finite. There are also **infinite intervals**. Here are some examples:

$[a, \infty)$		$x \ge a$
(a, ∞)		$x > a$
$(-\infty, b]$		$x \le b$
$(-\infty, b)$		$x < b$

DEFINITION

Let A and B be two sets of objects of any sort.

a) The set of all objects that are in ***both*** A and B is called ***A* intersection *B*** and is denoted $A \cap B$.

b) The set of all objects that are in ***either*** A or B ***or both*** is called ***A* union *B*** and is denoted $A \cup B$.

c) The set that contains no elements is called the **empty set** and is written as $\varnothing$.

These definitions can be applied to intervals, and they often are. Consider the following:

EXAMPLE 1

a) $(-\infty, 5) \cap (3, \infty)$ is the set of all numbers that are less than 5 and at the same time greater than 3. So it is the set $(3, 5)$.

b) Similarly, $(-\infty, 10] \cap (-1, \infty) = (-1, 10]$.

c) $(-\infty, 10) \cap (21, \infty) = \varnothing$, the empty set, since there are no numbers that are both less than 10 and at the same time greater than 21.

d) $(-\infty, 5) \cup (4, \infty) = (-\infty, \infty)$, the complete number line.

e) $(-\infty, 5) \cup (10, \infty)$ can't be written in a more compact form.

f) $(-\infty, 5) \cap [5, 10] = \varnothing$ ■

In e), we see the use of union to describe sets of points on the number line that are not connected. It is often the case that we speak of a set of numbers consisting of two or more intervals, finite or infinite. This is when the union sign really comes in handy!

1.8 Exercises

1) Represent the following sets of numbers using interval and number line notation:

a) $-1 \leq x \leq 3$

b) $-1 < x \leq 3$

c) $-3 \leq x < 1$

d) $-3 \leq x \leq 4$

e) $-\frac{1}{2} < x \leq \sqrt{2}$

f) $\pi \leq x \leq 5$

g) $x < -4$

h) $0 < x$

i) $3 \leq x$

j) $3 - \pi \leq x$

k) $x < 5$

l) $x \leq 3$

2) Represent the following sets of numbers using interval and number line notation:

a) $1 \leq x < 15$

b) $-2 \leq x \leq 2$

c) $-5 < x \leq -3$

3) Represent the following intervals using inequalities:

a) $(3, 7)$

b) $(-4, -1]$

c) $(-\infty, 19]$

d) $[2, 10)$

e) $[-2, -1]$

4) Represent the following intervals using inequalities:

a) $(3, 7]$

b) $[-23, 15]$

c) $(-\infty, 5]$

5) Simplify, if possible:

a) $(-\infty, 5) \cap [3, \infty)$

b) $(-\infty, 5) \cup [3, \infty)$

c) $(-\infty, -2) \cap [-2, \infty)$

d) $(-\infty, \infty) \cap [4, 7]$

e) $[3, 5] \cap (10, \infty)$

f) $(-\infty, 5] \cap [5, \infty)$

6) Sarah is trying to keep the water temperature in her chemistry experiment at 38°C. For the experiment to work, the relative error in the actual temperature must be less than 1%. Find the interval in which the actual temperature must lie, and write your solution using both inequalities and interval notation.