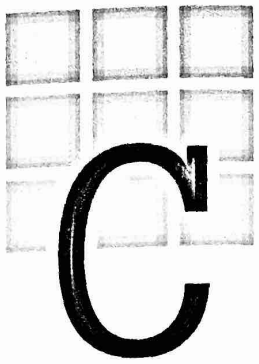
APPENDIX



Logarithmic Functions



Definition of Logarithms

Here's one more function you'll meet in science, engineering, and economics and sometimes in the newspaper. In symbols, it looks like this:

$$\log_a x$$

It is read as **log, to the base *a*, of *x*.** (**Do not** read this as "the log of a-to-the-x"!) There are several ways of defining it. Here's one.

DEFINITION

Let $a > 0$, $a \neq 1$. Then $\log_a x$ is **the number to which you raise *a* to get *x*.**

EXAMPLE 1 Demonstrate that $\log_2 8 = 3$.

Solution Here the base is 2 and $x = 8$. To what number do you have to raise 2 in order to get 8? Answer: 3, so $\log_2 8 = 3$. ■

EXAMPLE 2 Show that $\log_{10} 1{,}000{,}000 = 6$.

Solution Here the base is 10, and $x = 1{,}000{,}000$. What number do you have to raise 10 to in order to get 1,000,000 (6 zeros)? Answer: 6, so $\log_{10} 1{,}000{,}000 = 6$. ■

EXAMPLE 3 $\log_{10} 0.01 = ?$

Solution Here the base is 10, and $x = 0.01$. Write 0.01 as a power of 10. Here you go:

$$0.01 = \frac{1}{100} = \frac{1}{10^2} = 10^{-2}.$$

What number do you have to raise 10 to in order to get 10^{-2}? Answer: -2, of course, so $\log_{10} 0.01 = -2$. ■

EXAMPLE 4 $\log_2 32 = ?$

Solution Write 32 as a power of 2:

$$32 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^5,$$

so, $\log_2 32 = 5$. ■

EXAMPLE 5 $\log_3 81 = ?$

Solution Write 81 as a power of 3:

$$81 = 3 \cdot 3 \cdot 3 \cdot 3 = 3^4,$$

so, $\log_3 81 = 4$. ■

EXAMPLE 6 $\log_7 7^{15} = ?$

Solution 7^{15} is already written as a power of 7. What a silly question!
$\log_7 7^{15} = 15$ ■

EXAMPLE 7 $\log_a a^3 = ?$

Solution Another silly question! If you write a^3 as a power of a, obviously the exponent must be 3, so $\log_a a^3 = 3$. ■

Remark The function $\log_{10} x$ is known as the **common logarithm**. Sometimes you will see the expression $\log x$. Usually that means $\log_{10} x$.

C.1 Exercises

Evaluate the following:

1) $\log_9 81$

2) $\log_{11} 121$

3) $\log_2 \frac{1}{2}$

4) $\log_3 \frac{1}{81}$

5) $\log \frac{1}{1000}$

6) $\log \frac{1}{100}$

7) $\log_{1/2} 8$

8) $\log_{1/4} 64$

9) $\log_3 \sqrt{3}$

10) $\log_7 \sqrt{7}$

11) $\log_4 2\sqrt{2}$

12) $\log_b b^{13}$

13) $\log_a a^x$

14) $2^{\log_2 8}$

15) $10^{\log_{10}\left(\frac{1}{100}\right)}$

16) $3^{\log_3 81}$

17) $3^{\log_3 80}$

18) $10^{\log_{10}(1000)}$

19) $10^{\log_{10}(15)}$

20) $10^{\log_{10} x}$

21) $a^{\log_a x}$



Logs as Inverses of Exponential Functions

(Graphs and Equations)

You will see that there is a fundamental relationship between logs and exponentials: they are inverse to each other. Recall from Appendix B that f and g are called inverse to each other if all of the following are true:

1) $f(g(x)) = x$
2) $g(f(x)) = x$
3) domain of f = range of g
4) domain of g = range of f

Theorem

Let $a > 0$, $a \neq 1$. Then $\log_a x$ and a^x are inverse to each other.

Remark This theorem can be proved rigorously using the continuity properties from calculus, but that cannot be done here. However, if you let $f(x) = a^x$ and $g(x) = \log_a x$, you can examine the first two conditions.

a) $f(g(x)) = a^{g(x)} = a^{\log_a x}$. What does this mean? It is a, raised to the number to which you raise a to get x. So it equals x, that is

$$f(g(x)) = a^{g(x)} = a^{\log_a x} = x.$$

b) $g(f(x)) = \log_a f(x) = \log_a a^x = x$. (Just like Examples 6 and 7 in the previous section!)

So you haven't completely proved this theorem, but you can see that $\log_a x$ and a^x undo each other.

Knowing that $\log_a x$ and a^x are inverses allows you immediately to graph $\log_a x$. If you wish to graph the function $f(x) = \log_2 x$, you need only graph the function $g(x) = 2^x$ and flip it around the line $y = x$ (see Figure C.1).

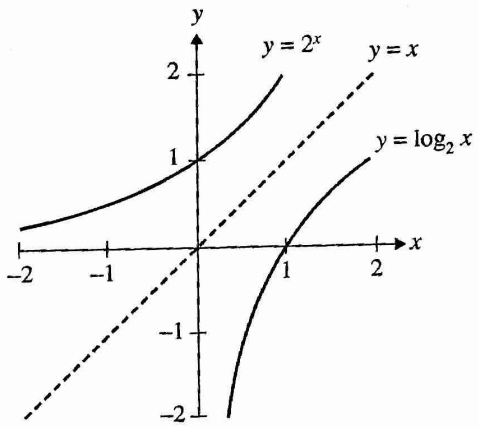


Figure C.1

Notice that the domain of $f(x) = \log_2 x$ is the set of all positive numbers, and the range is the set of all numbers. Notice also $\log_2 1 = 0$. Figure C.2 shows the common logarithm.

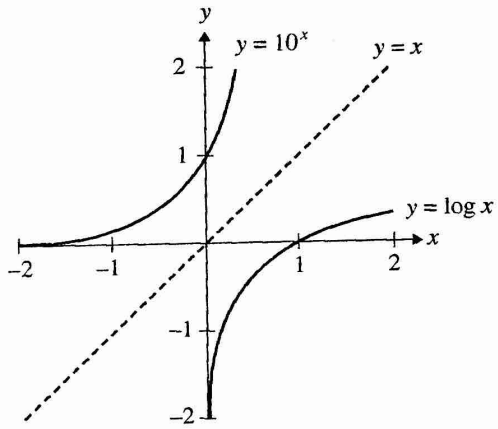


Figure C.2

The next example shows the logarithm for base $a = \frac{1}{2}$. First graph the function $\left(\frac{1}{2}\right)^x$, and then flip it about the line $y = x$ to obtain the graph of $f(x) = \log_{1/2} x$, as in Figure C.3.

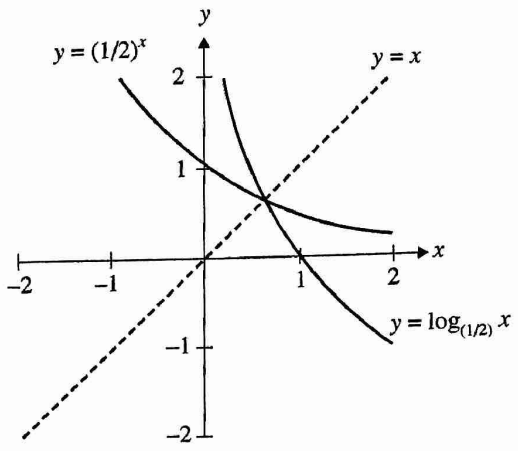


Figure C.3

Typically, you'll come across $\log_a x$ only for $a > 1$. Note that $\log_{1/a} x = -\log_a x$. You could get the previous graph by first graphing $\log_2 x$ and then rotating it about the x-axis! As with other functions, once you know what the log graph looks like, you can obtain other graphs by shifting and stretching. Figure C.4 shows how the graph of $\log_2(x + 3)$ is obtained from $\log_2 x$ by shifting it to the left three units.

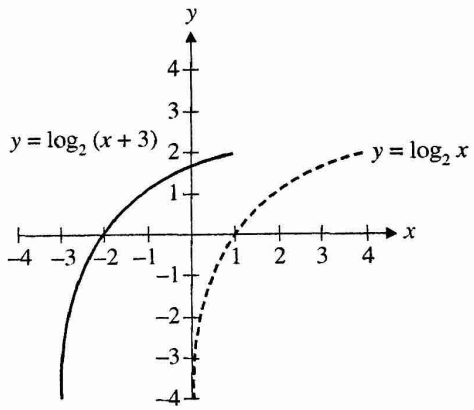


Figure C.4

By recalling the definition of logs and their relationship to exponentials, it is possible to solve some new equations containing such animals. The following examples show some of the problems you may encounter.

EXAMPLE 1 Solve $\log_2 x = 4$.

Solution We know that the function 2^x **undoes** the action of that log function. This means that if we apply the function 2^x to $\log_2 x$, we get x. (Recall that to apply the function $f(x)$ to the quantity a means to evaluate $f(x)$ at a, which gives us $f(a)$.)

Therefore, applying $f(x) = 2^x$ to $\log_2 x$ gives us $2^{\log_2 x}$, which we know is x, and applying $f(x) = 2^x$ to 4 gives us 2^4. So:

$$2^{\log_2 x} = 2^4$$

$$\therefore x = 16.$$

(The left side is x, because the log was "undone" by the action of 2^x; this is the true meaning of "inverseness," which is how $\log_2 x$ and 2^x are related.) ■

EXAMPLE 2 Solve $\log_{10} x = 3$.

Solution As in the last example, to "undo" the action of the log (now with base 10), apply 10^x to both sides. (Recall that applying 10^x to 3 means evaluating 10^x at 3.)

So $x = 10^3 = 1000$. ■

EXAMPLE 3 Solve $\log_{10}(x^2 - 4x + 14) = 1$.

Solution Again, apply 10^x to both sides. So,

$$(x^2 - 4x + 14) = 10^1 = 10,$$

or:

$$x^2 - 4x + 4 = 0$$

which factors as $(x - 2)^2 = 0$. So $x = 2$. ■

EXAMPLE 4 Solve $\log_3(x^2 - 3x - 7) = 1$.

Solution To "peel off" the log, apply 3^x, and in doing so you get:

$$(x^2 - 3x - 7) = 3^1 = 3,$$

or

$$x^2 - 3x - 10 = 0.$$

Factor this last equation to give:

$$(x - 5)(x + 2) = 0,$$

which has solutions $x = 5$ and $x = -2$. ■

EXAMPLE 5 Solve $2^{x^2+1} = 8$.

Solution Apply $\log_2 x$ to both sides (i.e., take $\log_2$ of both sides) and get

$$x^2 + 1 = \log_2 8,$$

or $$x^2 + 1 = 3.$$

This gives $$x^2 = 2,$$

or $$x = \pm\sqrt{2}.$$ ■

EXAMPLE 6 Solve $100^{\sin x} = 10$, for all $x \in \left[0, \frac{\pi}{2}\right]$.

Solution Apply $\log_{100} x$ to both sides.

Therefore $$\sin x = \log_{100} 10,$$

and since $$\log_{100} 10 = \frac{1}{2},$$

we have $$\sin x = \frac{1}{2},$$

or $$x = \frac{\pi}{6}.$$ ■

C.2 Exercises

1) a) Graph $f(x) = 2^{x+1}$.

   b) Show that $g(x) = \log_2 \frac{x}{2}$ is the inverse of $f(x)$ and graph it.

2) a) Graph $f(x) = 3^{x-1}$.

   b) Show that $g(x) = \log_3 3x$ is the inverse of $f(x)$ and graph it.

3) Solve the following:

   a) $2^{x-3} = 64$

   b) $3^{x+1} = 27$

   c) $4^{2x-3} = 16$

   d) $5^{x+5} = \frac{1}{125}$

4) Solve the following:

   a) $3^{x-2} = 27$

   b) $2^{x+1} = 64$

   c) $5^{2-x} = 125$

5) Solve $\log_3(x + 7) = -1$.

6) Solve $\log_3(x - 3) = 2$.

7) Solve $\log_{64} x^2 = \frac{1}{3}$.

8) Solve $\log_9 x^2 = \frac{1}{2}$.

9) Solve $\log_3(x^2 - 5) = 2$.

10) Solve $\log(\cos x) = 0$ for $x \in [0, 2\pi]$.

11) Using a calculator, graph $y = \log x$ between 0.5 and 2 by using several x values. Using the same x values, graph $y = \log(10x)$. What do you notice?



Laws of Logarithms

Your life among the logs is made much simpler when you know certain log laws. They're great when you're solving equations or simplifying expressions.

Log Laws:

Let $a > 0$, $a \neq 1$, and let $x > 0$ and $y > 0$. Then:

1) $\log_a xy = \log_a x + \log_a y$

2) $\log_a \frac{x}{y} = \log_a x - \log_a y$

3) $\log_a x^r = r \log_a x$

4) $\log_a 1 = 0$ for all a (But we knew that already!)

5) $\log_a x = \frac{\log_b x}{\log_b a}$, for any convenient b, for example, 10 or e. This "change of base" law can be a lifesaver if you can't handle $\log_a x$.

EXAMPLE 1 Solve $\log_2 x^2 + \log_2 2x = 4$.

Solution You can combine the left side using rule 1.

So $$\log_2 2x^3 = 4.$$

Now that the left side is a single log of an expression, you can apply 2^x to simplify it:

$$2x^3 = 2^4 = 16$$
$$x^3 = 8$$
$$x = 2.$$

You should check this solution:

$$\log_2(2^2) + \log_2(2 \cdot 2) = \log_2 4 + \log_2 4 = 2 + 2 = 4. \quad \blacksquare$$

EXAMPLE 2 Solve $\log_{10}(x^2 - 3x)^3 = 3$.

Solution Using the third law, you obtain

$$3 \log_{10}(x^2 - 3x) = 3,$$

or
$$\log_{10}(x^2 - 3x) = 1.$$

Now apply 10^x:
$$x^2 - 3x = 10^1 = 10.$$

Hence
$$x^2 - 3x - 10 = 0,$$
$$(x - 5)(x + 2) = 0,$$

and
$$x = 5 \text{ or } -2. \quad \blacksquare$$

In this example, you calculated two answers, both of which were valid because they satisfied the original equation. Sometimes, however, you may get some answers that are not valid. (They are called **extraneous solutions.** See Section 3.3) You must check your solutions to determine that they are valid. Consider the following.

EXAMPLE 3 Solve $\log_2 x + \log_2(x - 1) = 1$.

Solution Using the first law, you obtain

$\log_2 x + \log_2(x - 1) = \log_2(x(x - 1)) = \log_2(x^2 - x)$, which gives $\log_2(x^2 - x) = 1$. Now apply 2^x and get

$$x^2 - x = 2.$$

Hence
$$x^2 - x - 2 = 0$$

and
$$(x - 2)(x + 1) = 0$$

and
$$x = 2 \text{ or } -1.$$

However, when checking these "solutions," notice that the initial equation is not satisfied for $x = -1$, so $x = 2$ is the only solution. $\blacksquare$

EXAMPLE 4 Evaluate $\log_7 5$.

Solution You can't evaluate this directly. You could use a calculator, but it doesn't have $\log_7 x$. But, using the change of base law,

$$\log_7 5 = \frac{\log_{10} 5}{\log_{10} 7}$$

$$\cong \frac{0.699}{0.845} \cong 0.827. \quad \blacksquare$$

C.3 Exercises

1) Solve $\log_3 x + \log_3(x - 6) = 3$.
2) Solve $\log_2 x + \log_2(x - 2) = 3$.
3) Solve $\log y + \log y^2 = -1$.
4) Solve $\log y^2 + \log y^3 = -3$.
5) Solve $\log_6(2x + 1) - \log_6(2x - 1) = 1$.
6) Solve $\log_2 x^2 - \log_2(3x - 8) = 2$.
7) Find numbers a, x, and y where the value of $\log_a(x + y)$ is not equal to the value of $\log_a x + \log_a y$. (*Hint:* Try some values of a, x, and y where these expressions are easy to calculate. Maybe you'll get lucky.)
8) Approximate $\log_3 4$ using the change of base formula and your calculator.
9) Solve $2 \log_2(\sin x) + 1 = 0$ for x in $[0, 2\pi]$.
10) Solve $\log x - \log(x - 1) - 1 = 0$.
11) Solve $10^{\sin x} = 1$ for x in $[0, 2\pi]$.

C.4 The Natural Logarithm

In Appendix A, we introduced the exponential function e^x. Its inverse, $\log_e x$, is called the **natural logarithm.** For simplicity, $\log_e x$ is denoted by the symbol $\ln x$. To get its graph, we flip e^x about the line $y = x$, as in Figure C.5.

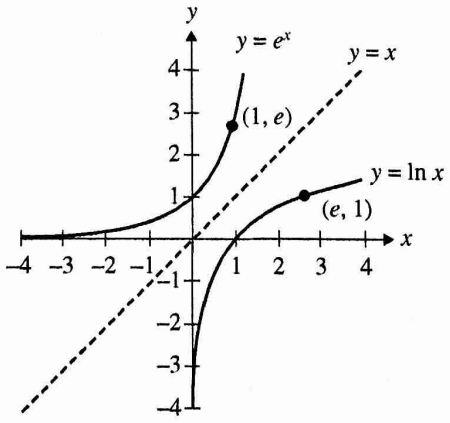


Figure C.5

Notice that $\ln 1 = 0$, and $\ln e = 1$. You will also encounter the natural logarithm when you study integration. It helps to know about $\ln x$ and how to graph related functions. Figure C.6 shows several horizontal shifts of the natural logarithm.

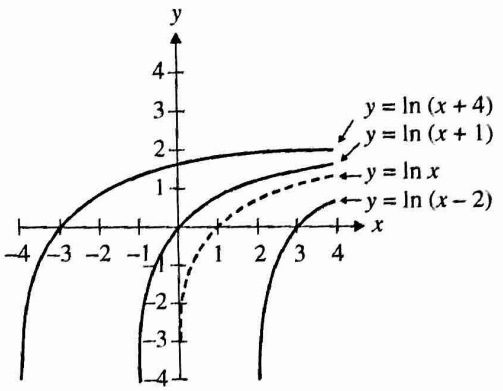


Figure C.6

In solving equations, you handle the natural log the same as any other log, remembering that the base is that special number e.

EXAMPLE 1 Solve $\ln(x^2 - 1)^3 = 1$.

Solution Apply e^x to get $(x^2 - 1)^3 = e$,

or $$x^2 - 1 = e^{1/3},$$

which can be solved to give

$$x = \pm\sqrt{1 + e^{1/3}}.$$

For most purposes you can leave your answer in this form. If its decimal approximation is needed, you can use your calculator to determine one. ■

EXAMPLE 2 Solve $e^{\sin x} = 1, x \in [0, 2\pi]$.

Solution Apply $\ln x$ to get $\sin x = \ln 1 = 0$,

and so $$x = 0, \pi, 2\pi.$$ ■

Remark As mentioned in Appendix A, any exponential a^x can be written in the form e^{kx} for some constant k. **THE TRICK:** write a as $e^{\ln a}$. So

$$a^x = (e^{\ln a})^x = e^{(\ln a)x}$$

(Recall: $(a^m)^n = a^{mn}$.)

That's it! Now consider the following.

EXAMPLE 3 Write 2^x in the form e^{kx}.

Solution $2^x = (e^{\ln 2})^x = e^{(\ln 2)x}$,

and since $\ln 2 \cong 0.693$,

$2^x \cong e^{0.693x}$. ■

EXAMPLE 4 Write $(0.345)^x$ in the form e^{kx}.

Solution $(0.345)^x = (e^{\ln 0.345})^x = e^{(\ln 0.345)x}$,

and since $\ln 0.345 \cong -1.06$,

$(0.345)^x \cong e^{-1.06x}$. ■

Remarks

a) If you convert the function a^x into the form e^{kx}, then if $a > 1$, as in Example 3, the constant $k > 0$. All functions of the form e^{kx}, for $k > 0$, look similar to the graph shown in Figure C.7.

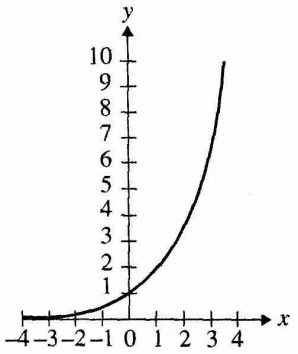


Figure C.7

In this case, the function is said to have **exponential growth.** An example from biology would be the growth of a colony of bacteria under ideal conditions.

b) On the other hand, if $0 < a < 1$, as in Example 4, then the constant $k < 0$. All functions of the form e^{kx}, for $k < 0$, look similar to the graph shown in Figure C.8.

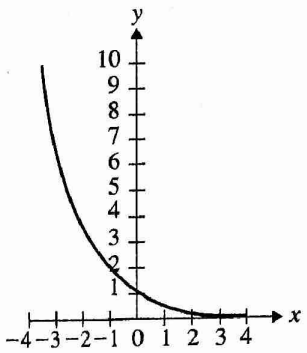


Figure C.8

In this case, the function is said to have **exponential decay.** In physics, the decay of radioactive substances is represented by such functions.

C.4 Exercises

1) Graph e^{x-1}.

2) Graph $e^{x+\pi}$.

3) Graph $y = \ln(x - 1)$.

4) Graph $y = 2 + \ln(x + 1)$.

5) Graph $y = 10 + \ln(x + 3)$.

6) Solve $\ln t - \ln t^2 = 5$.

7) Solve $\ln t + \ln t^2 = 6$.

8) Solve $e^{x^2+4x-5} = 1$.

9) Solve $e^{x^2+2x-3} = 1$.

10) Solve $e^{\ln(w^2+1)} = 5$. (Gift!)

11) Write 10^x and $(0.5)^x$ in the form e^{kx}. (Use a calculator.)

12) Find the inverse of $2e^{x-1}$.

13) Solve $2\ln(x + 1) - 1 = 0$.

14) Solve $\ln x - \ln \sqrt{x} - \frac{1}{2} = 0$.

15) The common logarithm is used to measure earthquake intensity on the Richter scale. The Richter scale describes the intensity, I, of an earthquake as the number $\log I - \log I_0$, where I_0 is the intensity of a small "benchmark" earthquake. Write the Richter scale rating as a single logarithm. What is the Richter scale rating for an earthquake with intensity $I = 100{,}000 I_0$?

16) Solve $7.2e^{0.18t} = 2$.

17) Solve $2 \ln x = 3.7$.

18) Solve $5.7 \ln 2x = 3.1$.

19) Solve $1.5e^{-2.3t} = 7.1$.

20) Solve $\pi e^{-\pi x} = 1$.