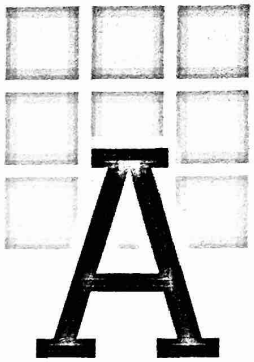
APPENDIX



Exponential Functions

A.1 Introduction

An **exponential function** has the form $f(x) = a^x$, where $a > 0$. The number a is called the **base**. Consider $a = 2$. It is clear what $f(x) = 2^x$ means for some values of x. For example,

$$f(0) = 2^0 = 1, \qquad f(1) = 2^1 = 2,$$
$$f(2) = 2^2 = 4 \qquad f(3) = 2^3 = 8,$$
$$f(-1) = 2^{-1} = \frac{1}{2}, \qquad f(-2) = 2^{-2} = \frac{1}{4},$$
$$f\left(\frac{1}{2}\right) = 2^{1/2} = \sqrt{2} \cong 1.414,$$

and
$$f(3.2) = 2^{3.2} = 2^3 \cdot 2^{1/5} = 8\sqrt[5]{2}.$$

This last one could be tough to calculate, but at least you know **what it means.** Use the preceding values to plot these points for the graph of $y(x) = 2^x$, shown in Figure A.1.

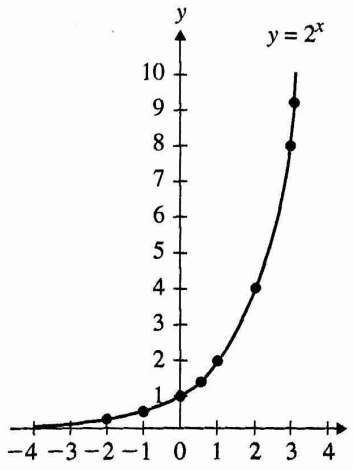


Figure A.1

We know what all rational exponents of 2 mean: $2^{m/n} = \left(\sqrt[n]{2}\right)^m$, if $\frac{m}{n}$ is in lowest terms. What happens at all the remaining (irrational) points? It can be shown (but not here) that there is exactly *one* smooth curve, always increasing, that can be drawn through all these points. That is the graph of 2^x, for x real. Notice that its domain is $(-\infty, \infty)$ and the range is $(0, \infty)$.

Graphs of a^x, for $a > 1$

Using these methods, plot the family of graphs a^x. Figure A.2 shows several exponential functions for the case when $a > 1$.

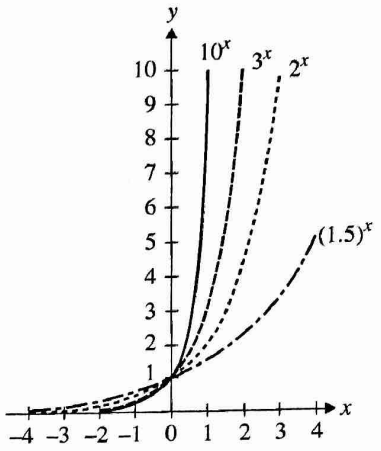


Figure A.2

Notes

a) As x gets large, each of these functions increases without bound (goes to ∞), but 10^x does it much faster than 3^x, etc.

b) As x goes to $-\infty$, these functions go to 0. But again, 10^x does it much faster than 3^x, etc.

c) All exponential functions pass through the point (0, 1).

Graphs of a^x, for $0 < a < 1$

Figure A.3 shows several exponential functions for the second case, where $0 < a < 1$, which can be obtained by plotting a few points. (A few thousand, that is, if you're the computer plotting this graph.)

By comparing Figures A.2 and A.3, it certainly **looks** as if $\left(\frac{1}{10}\right)^x$ is the mirror image of 10^x, $\left(\frac{1}{2}\right)^x$ is the mirror image of 2^x, etc. That this is true can be seen by a little

computation. Let $f(x) = 2^x$, and let $g(x) = \left(\frac{1}{2}\right)^x$. (Remember: $g(x)$ is the mirror image about the y-axis of $f(x)$, if and only if, $g(x) = f(-x)$.) Well, $g(x) = \left(\frac{1}{2}\right)^x = \frac{1^x}{2^x} = \frac{1}{2^x} = 2^{-x} = f(-x)$. This point is made graphically in Figure A.4.

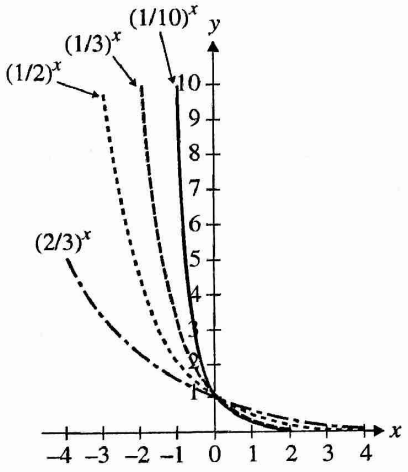


Figure A.3

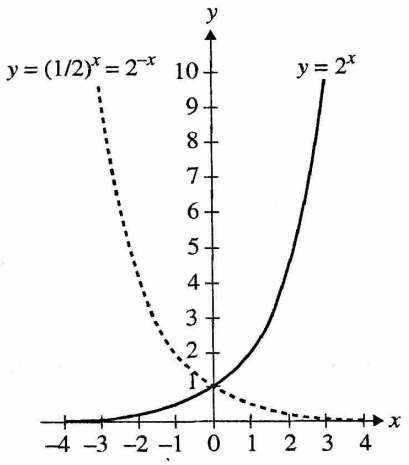


Figure A.4

By the way, notice that a^x is defined **for all** x only if $a > 0$. If $a < 0$, you can no longer have a^x for all x. For example, if $a = -1$, and $x = \frac{1}{2}$, we have $(-1)^{1/2} = \sqrt{-1}$, which is not a real number. So the family of functions a^x is defined for $a > 0$ and any real number x.

A.1 Exercises

1) Determine the behavior of the following exponential functions as $x \to \pm\infty$, then sketch the graph of the function, labeling at least three points:

a) $f(x) = \left(\frac{2}{3}\right)^x$ b) $f(x) = \left(\frac{3}{2}\right)^x$

c) $f(x) = 1.1^x$ d) $f(x) = 0.32^x$

2) Determine the behavior of the following exponential functions as $x \to \pm\infty$, then sketch the graph of the function, labeling at least three points:

a) $f(x) = \left(\frac{1}{3}\right)^x$ b) $f(x) = \left(\frac{5}{2}\right)^x$

3) Graph the function $f(x) = 2^x$, and then, using that result and the methods learned in Chapters 4 and 7, graph the following functions:

a) $f(x) = 2^{x-1}$ b) $f(x) = 2^{x+3}$

c) $g(x) = -2^{x+2}$ d) $f(x) = -2^{x-1}$

4) Graph the function $f(x) = 3^x$, and then, using that result, graph the following functions:

a) $f(x) = 3^{x+1}$ b) $f(x) = -\frac{1}{2} \cdot 3^{x+1}$

5) If $f(x) = \cos x$ and $g(x) = 3^x$, find:

a) $f(g(x))$ b) $g(f(x))$

c) $f(g(f(x)))$

6) If $f(x) = 2^x$ and $g(x) = \sin x$, find:

a) $f(g(x))$ b) $g(f(x))$

c) $f(g(f(x)))$

7) Use the methods of Section 8.2 to decompose the following functions (i.e., find $f(x)$ and $g(x)$ so that the given function is equal to $f(g(x))$, where $f(x)$ is the outermost function):

a) 2^{x-1} b) $3^{\sin x}$

c) $5^{\cos(x^2)}$

8) Decompose the functions (see above):

a) $\sin(3^x)$ b) $\sqrt{3^{x+2}}$

c) $2^{\sqrt{\sin x}}$

A.2 The Function e^x *(Also Called "The Exponential")*

Of all the exponential functions a^x, the one that has a 45° tangent at $x = 0$ is especially important. This exponential is depicted in Figure A.5.

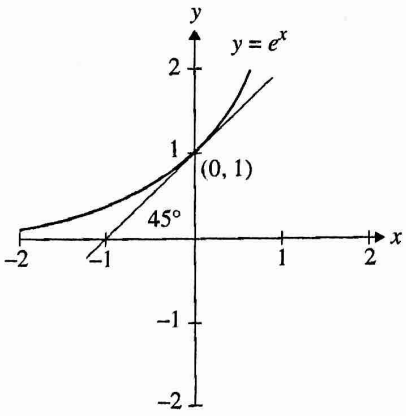


Figure A.5

You can tell from the graphs of 2^x and 3^x that this special function lies between them. The particular value of a that gives this exponential is called e. It can be calculated that $e \cong 2.718$. (In fact, $e \cong 2.718281828459045\ldots$) The exact reason why e^x is so important becomes clear in calculus, when you see that of all a^x, e^x has the simplest derivative.

Since any function a^x can be written as e^{kx}, for some k, that is the form ordinarily used in mathematics, science, and engineering. (See Appendix C for details.)

A.2 Exercises

1) Sketch:

a) $y = e^{x-1}$

b) $y = e^{-x}$

c) $y = -\frac{1}{2}e^{x}$

d) $f(x) = e^{x} - 1$

e) $y = -\frac{1}{2}e^{-x}$

2) Sketch:

a) $y = -e^{-x}$

b) $y = -2e^{-x}$

c) $y = e^{-x} + 1$

d) $y = 3 - e^{x}$

e) $y = 2 - 3e^{x}$

3) Sketch e^{-x^2}. (*Hint:* Plot and think!)

4) Sketch $y = e^x + e^{-x}$.

5) Let $f(x) = e^x$ and $g(x) = \tan x$. Find:

a) $f(g(x))$ b) $g(f(x))$

c) $\cos(f(f(x)))$

6) Let $f(x) = \sin x$ and $g(x) = e^{-x}$. Find:

a) $f(g(x))$ b) $g(f(x))$

c) $g(g(x))$

7) Simplify:

a) $(e^x)^3$ b) $\dfrac{e^{2x}}{e^x}$

c) $\dfrac{e^{2x} - 1}{e^x - 1}$

8) Simplify:

a) $(e^{-x})^2$ b) $\sqrt{e^{2x}}$

c) $\dfrac{e^x + 1}{e^{2x} - 1}$

9) Use the methods of Section 8.2 to decompose the following functions (i.e., find $f(x)$ and $g(x)$ so that the given function is equal to $f(g(x))$, where $f(x)$ is the outermost function):

a) e^{3x+1} b) $\cos(e^x)$

c) $\sin(e^{x^2})$ d) $\dfrac{1}{\cos(e^x)}$

e) $\cos^2(e^x)$

10) Decompose the following:

a) e^{-x} b) $\tan(e^{-x})$

c) $e^{-\tan x}$ d) $\sqrt{e^{\cos x}}$

e) e^{e^x}

11) Sketch $e^{-x}\cos x$.

12) Suppose a bacterial culture is growing such that it doubles its mass every 2 hr. If the culture begins as a mass of 1 g, write down a function that gives the mass at any time t. Graph this function. Under these conditions, what happens to the mass of the culture as $t \to \infty$?

13) Use your calculator to find:

a) $e^{2.15}$ b) e^{-10}

c) the value of x for which $e^x = 5$

14) Use your calculator to find:

a) e^3 b) the value of x for which $e^x = 0.1$

c) the value of x for which $2e^x + \sin x = 1$