Alternation Trading Proofs and Their Limitations

Sam Buss

Mathematical Foundations of Computer Science (MFCS)
IST Austria, Klosterneuburg
August 27, 2013

Fundamental problems for computer science include separating time classes from space classes, e.g.,

$$\mathrm{L} = \mathrm{P}? \qquad \text{and} \qquad \mathrm{P} = \mathrm{PSPACE}?$$

(L is log space; P is polynomial time.)
And, whether nondeterminism helps computation, e.g.,

$$\mathrm{P} = \mathrm{NP}?$$

Our primary successful tool for separating classes is diagonalization.

This talk: Limits of diagonalization for "L versus NP?"
Specifically: Alternation trading proofs as iterated diagonalization.

Towards separating logarithmic space (L) from non-deterministic polynomial time (NP).

$$\mathrm{L} \subseteq \mathrm{P} \subseteq \mathrm{NP} \subseteq \mathrm{PS\scriptstyle PACE} \subseteq \mathrm{E\scriptstyle XP}\mathrm{T\scriptstyle IME}.$$

Space hierarchy gives: $\mathrm{L} \neq \mathrm{PS\scriptstyle PACE}$.
Time hierarchy gives: $\mathrm{P} \neq \mathrm{E\scriptstyle XP}\mathrm{T\scriptstyle IME}$.
No other separations are known.

A series of results, especially since Fortnow [1997], has proved some *lower bounds* for the time complexity of sublinear space algorithms for Satisfiability ($\mathrm{S\scriptstyle AT}$) and thus for NP problems.

This talk discusses *upper bounds* on the *lower bounds* that can be obtained by present techniques of "alternation trading".

Barriers to separating L, P and NP include:

Oracle results: [Baker-Gill-Solovay, 1975] There are oracles collapsing the classes, so any proof of separation must not relativize.

Natural proofs: [Razborov-Rudich, 1997] Cryptographic assumptions imply that certain constructive separations are not possible.

Algebrization: [Aaronson-Wigderson, 2008] Proofs must not relativize to algebraic extensions of oracles.

Present talk: Bounds on the power of **alternation-trading** proofs for separating L and NP.

Alternation-trading proofs involve iterating the restricted space methods of Nepomnjasci [1970] together with simulations: essentially a sophisticated version of diagonalization.

Best alternation-trading results obtained so-far state that $\mathrm{S{\small AT}}$ is not computable in simultaneous time n^c and space n^ϵ for certain values of $c > 1$ and of $\epsilon > 0$. (But, not all such values!)

Theme: Better simulation methods give better diagonalization proofs for separating complexity classes.

Satisfiability

Definition (Satisfiability – $\textsc{Sat}$)

An instance of satisfiability is a set of clauses.
Each clause is a set of literals.
A *literal* is a negated or nonnegated propositional variable.
Satisfiability ($\textsc{Sat}$) is the problem of deciding if there is a truth assignment that sets at least one literal true in each clause.

Thm: Satisfiability is NP-complete.

Conjecture: Satisfiability is not polynomial time. ($\mathrm{P} \neq \mathrm{NP}$.)

Why is Satisfiability important?

1. Satisfiability is NP-complete.

2. Many other NP-complete problems are many-reducible to $\mathrm{S{\small AT}}$ in quasilinear time, that is, time $n \cdot (\log n)^{O(1)}$.

3. For a given non-deterministic machine M, the question of whether $M(x)$ accepts is reducible to $\mathrm{S{\small AT}}$ in quasilinear time. [sharpened Cook-Levin theorem].

Thus $\mathrm{S{\small AT}}$ is a "canonical" and natural non-deterministic time problem. Lower bounds on algorithms for $\mathrm{S{\small AT}}$ imply into the same lower bounds for many other NP-complete problems.

We always use the Random Access Memory (RAM) model for computation.
"DTIME" / "NTIME" = Deterministic/Nondeterministic time.

Theorem (Schnorr'78; Pippenger-Fischer'79; Robson'79,'91; Cook'88)

There is a $c > 0$ so that, for any language $L \in \mathrm{NTIME}(T(n))$, there is a quasi-linear time, many-one reduction to instances of SAT *of size $T(n)(\log T(n))^c$. In fact, each symbol of the instance of* SAT *is computable in polylogarithmic time $(\log T(n))^c$.*

Corollary

If $\mathrm{SAT} \in \mathrm{DTIME}(n^c)$, then $\mathrm{NTIME}(n^d) \subset \mathrm{DTIME}(n^{c \cdot d + o(1)})$.

The factor $n^{o(1)}$ hides logarithmic factors.

Definition

Let $c \geq 1$. $\mathrm{DTS}(n^c)$ is the class of problems solvable in simultaneous deterministic time $n^{c+o(1)}$ and space $n^{o(1)}$.

A series of results by Kannan [1984], Fortnow [1997], Lipton-Viglas, van Melkebeek, Williams, and others gives:

Theorem (R. Williams, 2007)

Let $c < 2\cos(\pi/7) \approx 1.8019$. *Then* $\textsc{Sat} \notin \mathrm{DTS}(n^c)$.

In this talk, we review these results and discuss a proof of their optimality relative to currently known proof techniques.

Nepomnjasci's method

Definition

$$^b(\exists n^c)^d \mathrm{DTS}(n^e)$$

denotes the class of problems taking inputs of length $n^{b+o(1)}$, existentially choosing $n^{c+o(1)}$ bits, keeping in memory a total of $n^{d+o(1)}$ bits (using time $n^{\max\{c,d\}+o(1)}$) which are passed to a deterministic procedure that uses time $n^{e+o(1)}$ and space $n^{o(1)}$.

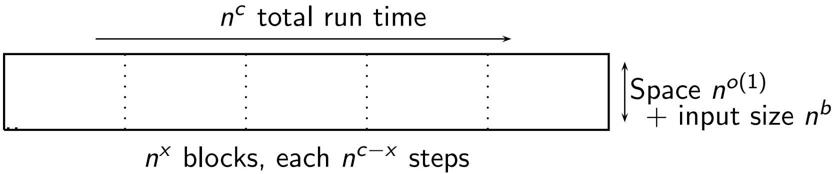
Theorem (by method of Nepomnjasci, 1970)

$^b\mathrm{DTS}(n^c) \subseteq {}^b(\exists n^x)^{\max\{b,x\}}(\forall n^0)^b\mathrm{DTS}(n^{c-x})$.

Proof next page....

${}^b\mathrm{DTS}(n^c) \subseteq {}^b(\exists n^x)^x(\forall n^0)^b\mathrm{DTS}(n^{c-x})$, for $x \geq b$

Proof idea: Split the n^c time computation into n^x many blocks.
Existentially guess the memory contents (apart from the input) at each block boundary (using $n^{x+o(1)}$ bits),
then universally choose one block to verify correctness (using $O(\log n) = n^{o(1)}$ universal choices),
and simulate that block's computation (in n^{c-x} time).



Alternation trading proofs [Williams]

An *alternation trading proof* is a proof that $\mathrm{SAT} \notin \mathrm{DTS}(n^c)$, for some fixed $c \geq 1$. It is a proof by contradiction, based on deducing

$$^1\mathrm{DTS}(n^a) \subseteq {}^1\mathrm{DTS}(n^b)$$

for some $a > b$, from the assumption that $\mathrm{SAT} \in \mathrm{DTS}(n^c)$.

The lines of an alternation trading proof are of the form

$$^1(\exists n^{a_1})^{b_2}(\forall n^{a_2})^{b_3} \cdots {}^{b_k}(Qn^{a_k})^{b_{k+1}}\mathrm{DTS}(n^{a_{k+1}}).$$

There are two kinds of inferences: "speedup" inferences that add quntifiers and reduce run time (based on Nepomnjascii) and "slowdown" inferences that remove a quantifier and increase run time (based on the S-P-F-R-C theorem)....

The rules of inferences for alternation trading proofs are:

Initial speedup: $(x \le a)$

$$^{1}\mathrm{DTS}(n^a) \subseteq {}^{1}(\exists n^x)^{\max\{x,1\}}(\forall n^0)^{1}\mathrm{DTS}(n^{a-x}),$$

Speedup: $(0 < x \le a_{k+1})$

$$\cdots {}^{b_k}(\exists n^{a_k})^{b_{k+1}}\mathrm{DTS}(n^{a_{k+1}}) \subseteq \cdots {}^{b_k}(\exists n^{\max\{x,a_k\}})^{\max\{x,b_{k+1}\}}(\forall n^0)^{b_{k+1}}\mathrm{DTS}(n^{a_{k+1}-x}),$$

Slowdown:

$$\cdots {}^{b_k}(\exists n^{a_k})^{b_{k+1}}\mathrm{DTS}(n^{a_{k+1}}) \subseteq \cdots {}^{b_k}\mathrm{DTS}(n^{\max\{cb_k,ca_k,cb_{k+1},ca_{k+1}\}}).$$

and the dual rules.

Example: alternation trading proof.

Let $1 < c < \sqrt{2}$. Then, if $\textsc{Sat} \in \mathrm{DTS}(n^c)$,

$$\begin{aligned} \mathrm{DTS}(n^2) &\subseteq (\exists n^1)^1(\forall n^0)^1\mathrm{DTS}(n^1) \\ &\subseteq (\exists n^1)^1\mathrm{DTS}(n^c) \\ &\subseteq \mathrm{DTS}(n^{c^2}). \end{aligned}$$

which is a contradiction. Proof uses a speedup-slowdown-slowdown pattern, also denoted **100**.

This proves:

Theorem (Lipton-Viglas, 1999)

$\textsc{Sat} \notin \mathrm{DTS}(n^{\sqrt{2}})$.

Better results can be found with more alternations.

Theorem (Fortnow, van Melkebeek, et. al)

$\mathrm{SAT} \notin \mathrm{DTS}(n^c)$, *where* $c < \phi \approx 1.618$, *the golden ratio.*

The optimal refutation with seven inferences derives:

Theorem (Williams)

$\mathrm{SAT} \notin \mathrm{DTS}(n^{1.6})$.

This proof uses the pattern of inferences: **1100100**, where "**1**" denotes a speedup and "**0**" denotes a slowdown.

Theorem (Williams)

Let $c < 2\cos(\pi/7) \approx 1.801$. Then $\mathrm{SAT} \notin \mathrm{DTS}(n^c)$.

This used proofs of the following $\mathbf{1}/\mathbf{0}$ patterns:

$$\mathbf{1}^n(\mathbf{10})^*(\mathbf{0}(\mathbf{10})^*)^n.$$

Based on using Maple to (unsuccessfully) search for better refutations, these were conjectured by Williams to be the best possible refutations.

We next discuss how to prove this conjecture, at least in the framework of currently known rules for alternation trading proofs.

Remark: If $\mathrm{SAT} \notin \mathrm{DTS}(n^c)$ for all c, then $\mathrm{L} \neq \mathrm{NP}$, something thought to be hard to prove.

$\mathrm{L} \subseteq \mathrm{NP} \subseteq \mathrm{P} \subseteq \mathrm{NP} \subseteq \mathrm{PSPACE}.$

Theorem (Buss-Williams)

There are alternation trading proofs of $\mathrm{SAT} \notin \mathrm{DTS}(n^c)$ for exactly the values $c < 2\cos(\pi/7)$.

Reduced alternation trading proofs

Two simplifications for a 'reduced" system:
1. Replace the superscripts "1" with "0".
2. Get rid of half the exponents! Replace each quantifier "$(Qn^{a_i})^{b_i}$" with just "Q^{b_i}".

The intuition is:

Firstly, that the values "1" can be made infinitesimal by making a_i's and b_i's large. Then the "1"s can be replaced by zeros.

Secondly, the a_i's are always dominated by the b_i's and thus are never important.

The simplified rules for alternation proofs become:

Initialization: $\quad {}^{0}\mathrm{DTS}(n^a) \vdash {}^{0}\exists^{0}\mathrm{DTS}(n^a).$

Speedup: $(0 < x \leq a)$

$$\cdots {}^{b_k}\exists^{b_{k+1}}\mathrm{DTS}(n^a) \vdash \; \cdots {}^{b_k}\exists^{\max\{x,b_{k+1}\}}\forall^{b_{k+1}}\mathrm{DTS}(n^{a-x}),$$

Slowdown: $\cdots {}^{b_k}\exists^{b_{k+1}}\mathrm{DTS}(n^a) \vdash \; \cdots {}^{b_k}\mathrm{DTS}(n^{\max\{cb_k, cb_{k+1}, ca\}}).$

Theorem

The reduced system has a refutation iff the original system has a refutation.

Approximate inference

Defn: Given Ξ and Ξ':

$$\Xi = {}^0\exists^{b_2}\forall^{b_3} \ldots {}^{b_k}Q^{b_{k+1}}\mathrm{DTS}(n^a)$$
$$\Xi' = {}^0\exists^{b'_2}\forall^{b'_3} \ldots {}^{b'_k}Q^{b'_{k+1}}\mathrm{DTS}(n^{a'}).$$

$\Xi \le \Xi'$ means $a \le a'$ and each $b_i \le b'_i$.

The *weakening rule* allows inferring Ξ' from Ξ; deduction with weakening is denoted $\Xi \overset{w}{\vdash} \Xi'$. The weakening rule does not add any power to the proof system.

Defn: $(\Xi + \epsilon)$ is obtained from Ξ by increasing a and each b_i by ϵ.

Definition (Approximate inference, $\Vdash$)

$\Xi \Vdash \Lambda$ if and only if for all $\epsilon > 0$ there exists a $\delta > 0$ such that

$$(\Xi + \delta) \overset{w}{\vdash} (\Lambda + \epsilon).$$

Achievability

Definition

Let $\mu \geq 1$ and $0 < \nu$. The pair $\langle \mu, \nu \rangle$ is *c-achievable* provided that, for all values a, b and d satisfying $c\mu b = \nu d$,

$$^a\exists^b \mathrm{DTS}(n^d) \Vdash {}^a\exists^{\mu b}\mathrm{DTS}(n^{\nu d}).$$

Theorem

If $\langle \mu, \nu \rangle$ is c-achievable for $\nu < 1/c$, then $\textsc{Sat} \notin \mathrm{DTS}(n^c)$.

Pf :

$$
\begin{array}{llll}
{}^0\mathrm{DTS}(n^1) & \vdash & {}^0\exists^0\mathrm{DTS}(n^1) & \text{Initialization} \\
 & \overset{w}{\vdash} & {}^0\exists^{\nu/(c\mu)}\mathrm{DTS}(n^1) & \text{Weakening} \\
 & \Vdash & {}^0\exists^{\nu/c}\mathrm{DTS}(n^\nu) & \text{By a } \langle \mu, \nu \rangle \text{ step} \\
 & \vdash & {}^0\mathrm{DTS}(n^{c\nu}) & \text{Slowdown}
\end{array}
$$

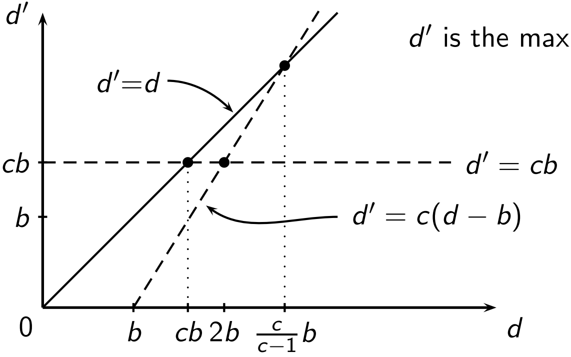
Note $c\nu < 1$. (Converse to proof holds too.)

Theorem

$\langle 1,\ c{-}1\rangle$ *is c-achievable with* $\mathbf{(10)^*}$ *derivations*

Pf. Let $\Xi = {}^a\exists^b\mathrm{DTS}(n^d)$, with $cb \le d$. Then

$$\Xi \vdash {}^a\exists^b\forall^b\mathrm{DTS}(n^{d-b}) \vdash {}^a\exists^b\mathrm{DTS}(n^{\max\{cb,c(d-b)\}}) = {}^a\exists^b\mathrm{DTS}(n^{d'}).$$



d' is the max of the dashed lines

"q.e.d."

Composition of c-achievable pairs

Theorem

Let $\langle\mu_1,\nu_1\rangle$ and $\langle\mu_2,\nu_2\rangle$ be c-achievable, with $c\nu_1\mu_2 \ge \mu_1$. Then $\langle\mu,\nu\rangle$ is c-achievable, where

$$\mu = c\nu_1\mu_2 \quad \text{and} \quad \nu = \frac{c\mu_1\nu_1\nu_2}{\mu_1+\nu_1\nu_2}.$$

Pf idea: Use a speedup, followed by a $\langle\mu_2,\nu_2\rangle$ step, then a slowdown, and finally a $\langle\mu_1,\nu_1\rangle$ step. If $c\nu_1\mu_2 < \mu_1$, then theorem holds with $\mu = \max\{c\nu_1\mu_2, \mu_1\}$ instead.

Theorem

The constructions above "subsume" all alternation trading proofs. There is an alternation trading proof of $\mathrm{SAT} \notin \mathrm{DTS}(n^c)$ iff an c-achievable pair with $\nu < 1/c$ can be constructed using the previous two theorems.

Understanding what is achievable

The expressions for μ and ν can be rewritten as:

$$\frac{1}{\mu} = \frac{1}{R}\left(\frac{1}{\mu_2}\right) \quad \text{and} \quad \frac{1}{\nu} = \frac{1}{T} - \frac{1}{R}\left(\frac{1}{T} - \frac{1}{\nu_2}\right).$$

where $\frac{1}{R} = \frac{1}{c\nu_1}$ and $\frac{1}{T} = \frac{\nu_1}{(c(\nu_1 - 1)\mu_1}$. Without loss of generality $\nu_1 > 1/c$ (otherwise we are done), and thus $\frac{1}{R} < 1$.

We think of $\langle \mu_1, \nu_1 \rangle$ as transforming $\langle \mu_2, \nu_2 \rangle$ to yield $\langle \mu, \nu \rangle$, and write this as

$$\langle \mu_1, \nu_1 \rangle : \langle \mu_2, \nu_2 \rangle \mapsto \langle \mu, \nu \rangle$$

This transformation makes μ_2 increase geometrically to get μ, and makes ν_2 contract inverse-geometrically towards T to get ν.

Define $\langle \mu_i, \nu_i \rangle$ by:

$$\begin{aligned} \langle \mu_0, \nu_0 \rangle &= \langle 1, c{-}1 \rangle, \\ \langle \mu_0, \nu_0 \rangle &: \langle \mu_i, \nu_i \rangle \mapsto \langle \mu_{i+1}, \nu_{i+1} \rangle. \end{aligned}$$

If

$$T_0 = \frac{(c\nu_0 - 1)\mu_0}{\nu_0} = \frac{c(c-1)-1}{c-1} < 1/c,$$

then some $\nu_i < 1/c$. This will give an alternation trading proof of $\mathrm{SAT} \notin \mathrm{DTS}(n^c)$. For $1 \le c \le 2$, this is equivalent to

$$\boxed{c^3 - c^2 - 2c + 1 < 0,}$$

i.e., $c < 2\cos(\pi/7)$.

This gives the desired alternation trading proof that $\mathrm{SAT} \notin \mathrm{DTS}(n^{2\cos(\pi/7)})$. [Williams]

The next theorem states $c = 2\cos(\pi/7)$ is the best possible. A key point is that the attraction points "T" only increase.

Lemma

If $\langle \mu_1, \nu_1 \rangle : \langle \mu_2, \nu_2 \rangle \mapsto \langle \mu, \nu \rangle$ *and if* $T_1 \geq 1/c$*, then* $T \geq T_2$*.*

Theorem

There are alternation trading proofs of $\mathrm{SAT} \notin \mathrm{DTS}(n^c)$ *for exactly the values* $c < 2\cos(\pi/7)$*.*

Time-Space Tradeoff Lower Bounds

Definition

$\mathrm{DTISP}(n^c, n^\epsilon)$ is the class of problems decidable in deterministic time $n^{c+o(1)}$ and space $n^{\epsilon+o(1)}$.

The notion of alternation trading proofs can be expanded to give proofs that $\mathrm{SAT} \notin \mathrm{DTISP}(n^c, n^\epsilon)$ for various values $1 \leq c < 2\cos(\pi/7)$ and $0 < \epsilon < 1$.

This is done by giving alteration trading proofs of

$$\mathrm{DTISP}(n^{\alpha c}, n^{\alpha\epsilon}) \subseteq \mathrm{DTISP}(n^{\beta c}, n^{\beta\epsilon})$$

for some $\alpha > \beta > 0$.

Rules of inference for DTISP

Initial speedup: $(e < x \leq a)$

$$^1\mathrm{DTISP}(n^a, n^e) \subseteq {}^1(\exists n^x)^{\max\{x,1\}}(\forall n^0)^{\max\{e,1\}}\mathrm{DTISP}(n^{a-x+e}, n^e)$$

Invoked only with $a = c \cdot e/\epsilon$.

Speedup: $(e < x \leq a_{k+1}.)$

$$\cdots {}^{b_k}(\exists n^{a_k})^{b_{k+1}}\mathrm{DTISP}(n^{a_{k+1}}, n^e)$$
$$\subseteq \quad \cdots {}^{b_k}(\exists n^{\max\{x,a_k\}})^{\max\{x,b_{k+1}\}}(\forall n^0)^{\max\{b_{k+1},e\}}\mathrm{DTISP}(n^{a_{k+1}-x+e}, n^e)$$

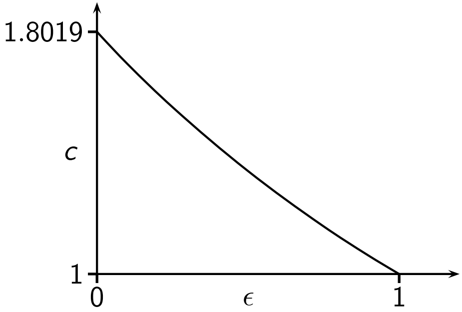
Slowdown: Let $a = \max\{b_k, a_k, b_{k+1}, a_{k+1}\}$.

$$\cdots {}^{b_k}(\exists n^{a_k})^{b_{k+1}}\mathrm{DTISP}(n^{a_{k+1}}, n^e) \subseteq \cdots {}^{b_k}\mathrm{DTISP}(n^{ca}, n^{\epsilon a}).$$

Based on extension of the theory of acheivable pairs to "acheivable triples", and on a computer-based search (C++), aided by theorems about pruning the searches:

Theorem [Buss-Williams] The following pairs are the optimal values c and ϵ for which there are alternating trading proofs that $\textsc{Sat} \notin \mathrm{DTISP}(n^c, n^\epsilon)$.

ϵ	c
0.001	1.80083
0.01	1.79092
0.1	1.69618
0.25	1.55242
0.5	1.34070
0.75	1.15765
0.9	1.06011
0.99	1.00583
0.999	1.00058



These values for c and ϵ are better than prior known lower bounds.

Open problems

- Find a closed form solution for the optimal $\mathrm{DTISP}(n^c, n^\epsilon)$ proofs. Even, find a simple characterization of how to construct the optimal proofs without resorting to a brute-force (pruned) search.
- There are many other flavors of alternation trading proofs, for instance for nondeterministic algorithms for tautologies. One could try giving proofs that the known alternation trading proofs are optimal.
- Most interesting: Try to find *new* principles that go beyond the presently known speedup and slowdown inferences, to give improved lower bound proofs.

Thank you!

ϵ	c	Number of Rounds	Number of Triples	Has Refutation
0.001	1.80084	7	167	No
	1.80083	11	455	Yes
0.01	1.79093	20	764	No
	1.79092	11	278	Yes
0.1	1.69619	248	3633	No
	1.69618	26	435	Yes
0.25	1.55242	249	2932	No
	1.55242	33	297	Yes
0.5	1.34071	203	1533	No
	1.34070	44	406	Yes
0.75	1.15766	155	1379	No
	1.15765	27	167	Yes
0.9	1.06012	146	454	No
	1.06011	19	88	Yes
0.99	1.00584	99	260	No
	1.00583	7	20	Yes
0.999	1.00059	3	3	No
	1.00058	24	10	Yes

ϵ	c	Number of Rounds	Number of Triples	Has Refutation
0.001	1.80084	7	167	No
	1.80083	11	455	Yes
0.01	1.79093	20	764	No
	1.79092	11	278	Yes
0.1	1.69619	248	3633	No
	1.69618	26	435	Yes
0.25	1.55242	249	2932	No
	1.55242	33	297	Yes
0.5	1.34071	203	1533	No
	1.34070	44	406	Yes
0.75	1.15766	155	1379	No
	1.15765	27	167	Yes
0.9	1.06012	146	454	No
	1.06011	19	88	Yes
0.99	1.00584	99	260	No
	1.00583	7	20	Yes
0.999	1.00059	3	3	No
	1.00058	24	10	Yes