Intuitionistic Validity in T-Normal Kripke Structures

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Abstract

Let T be a first-order theory. A T-normal Kripke structure is one in which every world is a classical model of T. This paper gives a characterization of the intuitionistic theory $\mathcal{H}T$ of sentences intuitionistically valid (forced) in all T-normal Kripke structures and proves the corresponding soundness and completeness theorems. For Peano arithmetic (PA), the theory $\mathcal{H}PA$ is a proper subtheory of Heyting arithmetic (HA), so HA is complete but not sound for PA-normal Kripke structures.

1 Introduction

A Kripke structure for first-order intuitionistic logic may be viewed as a set of classical first-order structures, called *worlds*, which are partially ordered by a reachability relation. If a world $\mathcal{M}_2$ is reachable from $\mathcal{M}_1$, then $\mathcal{M}_1$ is

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embedded in $\mathcal{M}_2$ so that each atomic formula which holds in $\mathcal{M}_1$ also holds in $\mathcal{M}_2$. The interpretation of equality in a world need not be true equality; hence it is permissable that two non-equal elements of a world are equal in a reachable world.

For $\mathcal{M}$ a world in a Kripke structure and φ a first-order formula, the notion of φ being intuitionistically true or "forced" at $\mathcal{M}$ is defined inductively on the complexity of φ. For example, $\neg\psi$ is forced at $\mathcal{M}$ if ψ is not forced at any world reachable from $\mathcal{M}$. (We give the definition in full below.) One can also regard the *classical* truth or falsity of φ in $\mathcal{M}$ by ignoring the rest of the worlds and just considering the usual classical (Tarskian) semantics of truth in the structure $\mathcal{M}$.

If T is a classical first-order theory in a language $\mathcal{L}$, then a T-normal Kripke structure is a Kripke structure for intuitionistic logic in the language $\mathcal{L}$ in which each world classically satisfies T. The first-order sentences which are intuitionistically valid (or, *forced*) in all T-normal Kripke structures clearly form an intuitionistic theory, denoted $\mathcal{H}T$. The main result of this paper is to give an axiomatization of $\mathcal{H}T$; namely, by

$$\{\theta^\varphi \to \varphi : \theta \text{ is semipositive and } T \vdash_c \neg\theta\}$$

(Complete definitions are given below.) We examine carefully the case of Peano and Heyting arithmetic (PA and HA) in the language of primitive recursive arithmetic (PRA) and show that HA properly contains $\mathcal{H}PA$ and thus HA is complete, but not sound, for PA-normal Kripke structures.

From a philosophical point of view, it may seem rather strange to consider the intuitionistic theory of T-normal Kripke structures since this involves a mixing of the notions of classical truth and intuitionistic truth. However, this does make sense if one takes the view that intuitionistic logic is be interpreted in the setting of (possible) states of knowledge or in the setting of possible states of databases. For example, Nerode [3] motivates intuitionist logic as modeling assertions about the possible contents of a database at various times. For this, one may have a set of constraints classically true for any possible database; the constraints form the classical theory T. The theory $\mathcal{H}T$ is then the sentences which are intuitionistically valid for Kripke structures of databases satisfying the constraints of T.

Another reason for studying T-normal Kripke structures is that it allows model-theoretic techniques for classical logic to be applied to intuitionistic logic. This was the author's original motivation and has already been applied in Buss [1] where the model-theoretic proof of an independence result for the classical theory PV_1 due to Krajíček and Pudlák was used to reprove and strengthen an independence result due to Cook and Urquhart for the intuitionistic theory IPV. Similarly, to prove that a formula is an intuitionistic consequence of HA, it suffices to show that it is forced in all PA-normal Kripke structures.

For Heyting arithmetic, the notion of PA-normal Kripke models has already been considered by van Dalen, et. al. [5]—they called such models "locally PA". They show that any model of Heyting arithmetic on a finite frame is PA-normal and that models of Heyting arithmetic on a frame of order type ω contain infinitely many classical models of PA. These results seem neither to imply nor be implied by the results of this paper.

2 Definitions

A *classical model* for classical first-order logic is defined as usual, using Tarskian semantics. We adopt the convention, however, that equality ($=$) in a classical model is interpreted and may not be the true equality; of course, the interpretation of equality must be an equivalence relation that respects the functions and relations. The corresponding semantic notion for intuitionistic first-order logic is that of a Kripke model. We briefly review the definition of Kripke structures as models for intuitionistic logic; see Troelstra and van Dalen's textbook [4] for a thorough treatment.

Fix a first-order language $\mathcal{L}$. A Kripke structure $\mathcal{K}$ for the language $\mathcal{L}$ is an ordered pair $(\{\mathcal{M}_i\}_{i\in\mathcal{I}}, \preccurlyeq)$ where $\{\mathcal{M}_i\}_{i\in\mathcal{I}}$ is a set of (not necessarily distinct) classical structures for the language $\mathcal{L}$ indexed by elements of the set $\mathcal{I}$ and where $\preccurlyeq$ is a reflexive and transitive binary relation on $\mathcal{I}$. The $\mathcal{M}_i$'s are called *worlds*. To improve readability, we shall usually write $\mathcal{M}_i \preccurlyeq \mathcal{M}_j$ instead of $i \preccurlyeq j$ even though this is an abuse of notation as the $\mathcal{M}_i$'s may not be distinct. $\mathcal{K}$ must satisfy the property that whenever $\mathcal{M}_i \preccurlyeq \mathcal{M}_j$ then $\mathcal{M}_i$ is a "weak substructure" of $\mathcal{M}_j$ in that the domain $|\mathcal{M}_i|$ of $\mathcal{M}_i$ is a subset of the domain of $\mathcal{M}_j$ and for all atomic formulas $\varphi(x_1,\ldots,x_k)$ and all $a_1,\ldots a_k \in |\mathcal{M}_i|$, if $\mathcal{M}_i \vDash \varphi(a_1,\ldots,a_k)$ then also $\mathcal{M}_j \vDash \varphi(a_1,\ldots,a_k)$. Note that this allows unequal elements in $|\mathcal{M}_i|$ to become equal in $|\mathcal{M}_j|$.

If φ is a formula and if $\vec{c} \in |\mathcal{M}_i|$ then we define $\mathcal{M}_i \models \varphi(\vec{c})$, $\mathcal{M}_i$ *classically* *satisfies* $\varphi(\vec{c})$, as usual, ignoring the rest of the worlds in the Kripke structure. To define the intuitionistic semantics, $\mathcal{M}_i \Vdash \varphi(\vec{c})$, $\mathcal{M}_i$ *forces* $\varphi(\vec{c})$, is defined inductively on the complexity of φ as follows:[†]

(1) If φ is atomic, $\mathcal{M}_i \Vdash \varphi$ if and only if $\mathcal{M}_i \models \varphi$. As a special case, $\mathcal{M}_i \nVdash \bot$.

(2) If φ is $\psi \wedge \chi$ then $\mathcal{M}_i \Vdash \varphi$ if and only if $\mathcal{M}_i \Vdash \psi$ and $\mathcal{M}_i \Vdash \chi$.

(3) If φ is $\psi \vee \chi$ then $\mathcal{M}_i \Vdash \varphi$ if and only if $\mathcal{M}_i \Vdash \psi$ or $\mathcal{M}_i \Vdash \chi$.

(4) If φ is $\psi \to \chi$ then $\mathcal{M}_i \Vdash \varphi$ if and only if for all $\mathcal{M}_j \succcurlyeq \mathcal{M}_i$, if $\mathcal{M}_j \Vdash \psi$ then $\mathcal{M}_j \Vdash \chi$.

(5) If φ is $(\exists x)\psi(x)$ then $\mathcal{M}_i \Vdash \varphi$ if and only if there is some $b \in |\mathcal{M}_i|$ such that $\mathcal{M}_i \Vdash \psi(b)$.

(6) If φ is $(\forall x)\psi(x)$ then $\mathcal{M}_i \Vdash \varphi$ if and only if for all $\mathcal{M}_j \succcurlyeq \mathcal{M}_i$ and all $b \in |\mathcal{M}_j|$, $\mathcal{M}_j \Vdash \varphi(b)$.

An immediate consequence of the definition of forcing is that if $\mathcal{M}_i \Vdash \varphi$ and $\mathcal{M}_i \preccurlyeq \mathcal{M}_j$ then $\mathcal{M}_j \Vdash \varphi$; this is proved by induction on the complexity of φ.

We shall use $\neg\psi$ as an abbreviation for $\psi \to \bot$. Hence, $\mathcal{M}_i \Vdash \neg\psi$ if and only if for all $\mathcal{M}_j \succcurlyeq \mathcal{M}_i$, $\mathcal{M}_j \nVdash \psi$.

A formula $\varphi(\vec{x})$ is *valid* in $\mathcal{K}$, denoted $\mathcal{K} \Vdash \varphi(\vec{x})$, if and only if for all worlds $\mathcal{M}_i$ and all $\vec{c} \in |\mathcal{M}_i|$, $\mathcal{M}_i \Vdash \varphi(\vec{c})$. A set of formulas Γ is valid in $\mathcal{K}$, $\mathcal{K} \Vdash \Gamma$, if and only if every formula in Γ is valid in $\mathcal{K}$. We define that $\Gamma \Vdash \varphi$, φ is a *Kripke consequence* of Γ, if and only if for every Kripke structure $\mathcal{K}$, if $\mathcal{K} \Vdash \Gamma$ then $\mathcal{K} \Vdash \varphi$.

A theory is, by definition, a set of sentences closed under logical implication. (Formulas may contain free variables whereas sentences may not.) If Γ is an arbitrary set of sentences, we write $\Gamma \vdash_i \varphi$ and $\Gamma \vdash_c \varphi$ to indicate that the formula φ is intuitionistically (respectively, classically) provable from the sentences in Γ. A set of sentences T is an intuitionistic (resp., classical) theory if and only if T is closed under intuitionionistic (resp., classical) provability. Hence, $T \vdash_i \varphi$ (resp., $T \vdash_c \varphi$) if and only if the universal closure of φ is in T.

Let Γ be a set of sentences and φ a formula. The strong soundness and completeness theorems for classical logic state that $\Gamma \vdash_c \varphi$ if and only

[†]A more proper notation would be $(\mathcal{K}, \mathcal{M}_i) \Vdash \varphi(\vec{c})$ or even $(\mathcal{K}, i) \Vdash \varphi(\vec{c})$ but we use the simpler notation $\mathcal{M}_i \Vdash \varphi(\vec{c})$ when $\mathcal{K}$ is specified by the context.

if $\Gamma \models \varphi$. Likewise, the strong soundness and completeness theorems for intuitionistic logic and Kripke structures state that $\Gamma \vdash_i \varphi$ if and only if $\Gamma \Vdash \varphi$.

Definition Let T be a classical theory. A Kripke structure $\mathcal{K}$ is said to be T*-normal* if and only if, for each world $\mathcal{M}$ of $\mathcal{K}$, M classically satisfies T (i.e., $M \models T$).

Definition A formula θ is *semipositive* if and only if each subformula of θ of the form $\theta_1 \to \theta_2$ has θ_1 atomic. Since $\neg\psi$ abbreviates $\psi \to \bot$, this implies that any negated subformula of θ is atomic.

Definition Let θ and φ be formulas. The formula θ^φ is obtained from θ by replacing each atomic subformula χ of θ with $(\chi \vee \varphi)$.

In forming θ^φ, bounded variables in θ are renamed as necessary so that no free variable of φ is bound in θ. The definition of θ^φ is originally due to H. Friedman [2] who called it the φ*-translation of* θ.

Definition Let T be a classical theory. Then $\mathcal{H}T$ is the intuitionistic theory axiomatized by the (universal closures of the) formulas

$$\{\theta^\varphi \to \varphi : \theta \text{ is a semipositive formula, } T \vdash_c \neg\theta\text{, and } \varphi \text{ is any formula.}\}$$

Note that if θ is atomic and $T \vdash_c \theta$ then $\mathcal{H}T \vdash_i \theta$. To see this, note that $\neg\theta$ is semipositive and $T \vdash_c \neg\neg\theta$. Now, the formula $(\neg\theta)^\theta$ is $(\theta \vee \theta) \to (\bot \vee \theta)$ since $\neg\theta$ abbreviates $\theta \to \bot$. So $\mathcal{H}T$ has the axiom $(\neg\theta)^\theta \to \theta$ which obviously intuitionistically implies θ. Likewise, if θ is atomic and $T \vdash_c \neg\theta$ then $\mathcal{H}T \vdash_i \neg\theta$. To see this use the axiom $\theta^\bot \to \bot$ of $\mathcal{H}T$.

Both of the latter observations are corollaries of the completeness theorem proved in the next section. The soundness theorem further implies the converses; namely, if θ is atomic or negated atomic and if $\mathcal{H}T \vdash_i \theta$ then $T \vdash_c \theta$. In fact the soundness theorem implies that $\mathcal{H}T \subseteq T$.

It is easy to see that $\varphi \to \theta^\varphi$ is intuitionistically valid for all formulas φ and θ. Hence $\mathcal{H}T \vdash_i \theta^\varphi \leftrightarrow \varphi$ for all semipositive θ such that $T \vdash_c \neg\theta$.

3 Soundness and Completeness

We next establish the soundness and completeness theorems for the intuitionistic theory $\mathcal{H}T$ with respect to T-normal Kripke structures. We treat the soundness theorem first since its proof is by far the simpler.

Theorem 1 (Soundness Theorem) *Let T be a classical theory. If $\mathcal{H}T \vdash_i \psi$ and if $\mathcal{K}$ is a T-normal Kripke model, then $\mathcal{K} \Vdash \psi$.*

Proof It clearly suffices to prove the soundness theorem for the case where ψ is an axiom of $\mathcal{H}T$; i.e., when ψ is a formula $\theta^\varphi \to \varphi$ where θ is semipositive and $T \vdash_c \neg\theta$. If $\mathcal{M}$ is a classical structure, we define an $\mathcal{M}$-formula to be a formula in the language of T plus constant symbols for members of the universe of $\mathcal{M}$. To prove the Soundness Theorem, it will suffice to prove that if $\mathcal{M}$ is a world in a T-normal Kripke structure $\mathcal{K}$, θ and φ are $\mathcal{M}$-sentences, θ is semipositive and $T \vdash_c \neg\theta$ then $\theta^\varphi \to \varphi$ is intuitionistically forced true in $\mathcal{M}$. To prove this we need the following claim:

Claim: Suppose $\mathcal{K}$ is a Kripke structure, $\mathcal{M}$ is a world in $\mathcal{K}$, θ is a semipositive $\mathcal{M}$-sentence and φ is a $\mathcal{M}$-sentence. If $\mathcal{M} \not\Vdash \varphi$ and $\mathcal{M} \Vdash \theta^\varphi$ then $\mathcal{M} \models \theta$.

The Soundness Theorem follows immediately from the claim since every world in $\mathcal{K}$ is classically satisfies T and hence $\neg\theta$.

The proof of the claim is by induction on the logical complexity of θ. Suppose $\mathcal{M} \Vdash \theta^\varphi$ but $\mathcal{M} \not\Vdash \varphi$. If θ is atomic, then θ^φ is $\theta \vee \varphi$ and the claim is obvious from the definition of forcing. If θ is a formula of the form $\theta_1 \vee \theta_2$ then θ^φ is $\theta_1^\varphi \vee \theta_2^\varphi$ and $\mathcal{M} \Vdash \theta^\varphi$ implies that $\mathcal{M} \Vdash \theta_1^\varphi$ or $\mathcal{M} \Vdash \theta_2^\varphi$. And by the induction hypothesis, $\mathcal{M} \models \theta_1$ or $\mathcal{M} \models \theta_2$ and hence $\mathcal{M} \models \theta$. The case where θ is a conjunction is handled similarly. If θ is of the form $(\exists x)\theta_1(x)$, then there is some $m \in |\mathcal{M}|$ such that $\mathcal{M} \Vdash \theta_1^\varphi(m)$ so by the induction hypothesis $\mathcal{M} \models \theta_1(m)$ and hence $\mathcal{M} \models \theta$. The case where θ begins with a universal quantifier is similar and uses the fact that θ^φ is forced in any world reachable from $\mathcal{M}$. Finally, suppose θ is a formula of the form $\theta_1 \to \theta_2$. Since θ is semipositive, θ_1 must be atomic; thus θ^φ is the formula $(\theta_1 \vee \varphi) \to (\theta_2^\varphi)$. Since θ_1 is atomic, $\mathcal{M} \Vdash \theta_1$ if and only if $\mathcal{M} \models \theta_1$. Now if $\mathcal{M} \not\Vdash \theta_1$ then clearly $\mathcal{M} \models \theta$. On the other hand, if $\mathcal{M} \Vdash \theta_1$ then $\mathcal{M} \Vdash \theta_2^\varphi$; hence, by the induction hypthesis, $\mathcal{M} \models \theta_2$ and, again, $\mathcal{M} \models \theta$.
Q.E.D. Soundness Theorem

Theorem 2 (Completeness Theorem) *Let T be a classical theory and φ be any sentence. If $\mathcal{H}T \nvdash_i \varphi$ then there is a T-normal Kripke structure $\mathcal{K}$ such that $\mathcal{K} \nVdash \varphi$.*

Together, Theorems 1 and 2 show that $\mathcal{H}T$ is the intuitionistic theory of T-normal Kripke structures.

The proof of Theorem 2 takes the rest of this section and will proceed along the lines of the proof of the usual strong completeness theorem for intuitionistic logic as exposited in section 2.6 of Troelstra and van Dalen [4]. The new ingredient and the most difficult part in our proof is Lemma 4 which ensures that the Kripke structure we construct is T-normal.

We let T be a fixed classical theory for the rest of this section. For simplicity, we assume that T and its language are countable; however, the proof can be readily extended to uncountable languages.

Definition Let C be a set of constant symbols. A C-formula or C-sentence is a formula or sentence in the language of T plus constant symbols in C. All sets of constants are presumed to be countable.

Definition A set of C-sentences Γ is *C-saturated* provided the following hold:

(1) Γ is intuitionistically consistent,

(2) For all C-sentences φ and ψ, if $\Gamma \vdash_i \varphi \vee \psi$ then $\Gamma \vdash_i \varphi$ or $\Gamma \vdash_i \psi$.

(3) For all C-sentences $(\exists x)\varphi(x)$, if $\Gamma \vdash_i (\exists x)\varphi(x)$ then for some $c \in C$, $\Gamma \vdash_i \varphi(c)$.

The next, well-known lemma shows that C-saturated sets can be readily constructed.

Lemma 3 *Let Γ be a set of sentences and φ be a sentence such that $\Gamma \nvdash_i \varphi$. If C is a set of constant symbols containing all constants in Γ plus countably infinitely many new constant symbols, then there is a C-saturated set Γ^* containing Γ such that $\Gamma^* \nvdash_i \varphi$.*

The proof of Lemma 3 is quite simple, merely enumerate with repetitions all C-sentences which either begin with an existential quantifier or are a disjunction and then form Γ^* by adding new sentences to Γ so that (2) and (3) of the definition of C-saturated are satisfied. This can be done so that φ is still not an intuitionistic consequence. For example, if during the construction of Γ^* we enumerate a formula $(\exists x)\psi(x)$ which is an intuitionistic consequence of the sentences already in Γ^* then we can pick any constant c from C that does (yet) occur in Γ^* or in φ and put the formula $\psi(c)$ into Γ^*. This will preserve the property that $\Gamma^* \nvdash_i \varphi$. (For a full proof, refer to lemma 2.6.3 of [4].) $\Box$

In the proof of the usual completeness theorem for Kripke models and intuitionistic logic, the C-saturated sets of sentences constructed with Lemma 3 specify worlds in a canonical Kripke model. However, we need to construct a Kripke model which is T-normal and Lemma 3 does not provide us with sets Γ^* that specify worlds that classically satisfy T. Instead, we need to establish the harder Lemma 4 below.

A C-saturated set Γ defines a world with domain C in which an atomic formula φ is forced if and only $\Gamma \vdash_i \varphi$. We shall also think of Γ specifying a classical structure $\mathcal{M}_\Gamma$ defined as follows:

Definition Suppose Γ is a C-saturated set. Then $\mathcal{M}_\Gamma$ is defined to be the classical structure in the language of T plus constant symbols in C such that the domain of $\mathcal{M}_\Gamma$ is C itself (so $c^{\mathcal{M}_\Gamma} = c$) and such that for every atomic C-sentence φ, $\mathcal{M}_\Gamma \vDash \varphi$ if and only if $\Gamma \vdash_i \varphi$.

It is straightforward to check that the definition of $\mathcal{M}_\Gamma$ does indeed specify a unique classical structure. The first thing to check is that the function and constant symbols of T can be uniquely interpreted. This follows from the fact that if t is a closed term then $(\exists x)(x = t)$ is intuitionistically valid and hence, by C-saturation, there is some constant $c \in C$ such that $\Gamma \vdash_i c = t$. The second thing to notice is that equality is interpreted in $\mathcal{M}_\Gamma$ (i.e., $=^{\mathcal{M}_\Gamma}$ may not be true equality) since there may be distinct $c_1, c_2 \in C$ such that $\Gamma \vdash_i c_1 = c_2$. It is easy to check that the requisite equality axioms hold (it suffices to do this for atomic formulas).

In order to prove Theorem 2 we must construct sets Γ so that the structures $\mathcal{M}_\Gamma$ are classical models of T; Lemma 4 is the crucial tool for this:

Lemma 4 *Suppose* Γ *is a set of* C*-sentences,* φ *is a* C*-sentence and* $\Gamma \supseteq \mathcal{H}T$*. Further suppose* $\Gamma \nvdash_i \varphi$*. Then there is a set* Γ^* *of sentences and a set* C^* *of constants such that*

(a) $\Gamma^* \supseteq \Gamma$

(b) Γ^* *is* C^**-saturated*

(c) $\Gamma^* \nvdash_i \varphi$*, and*

(d) $\mathcal{M}_{\Gamma^*} \vDash T$*.*

Note that if condition (d) were omitted, Lemma 4 would be equivalent to Lemma 3.

Proof Γ^* and $\mathcal{M}_{\Gamma^*}$ are constructed by a technique similar to Henkin's proof of Gödel's completeness theorem. We pick C^* to be C plus countably infinitely many new constant symbols and enumerate the C^* formulas as $\alpha_1, \alpha_2, \alpha_3, \ldots$ with each C^*-formula appearing infinitely many times in the enumeration. We shall form classically consistent sets of sentences $\Pi_0, \Pi_1, \Pi_2, \ldots$ so that $\Pi_0 \supseteq T$ and so that, for all k, $\Pi_k \supseteq \Pi_{k-1}$ and so that, for any C^*-formula α_j, either α_j or $\neg\alpha_j$ eventually is put into a Π_k. Furthermore, if $\alpha_j = (\exists x)\beta(x)$ and α_j enters a Π_k then for some constant symbol c, $\beta(c)$ will be in some $\Pi_{k'}$. Thus, as usual in a Henkin-style model construction, the union Π^* of the Π_k's will specify a classical model $\mathcal{M}$ of T with domain C^*.

While defining the sets Π_k we also define sets Γ_k and C_k so that

$$\Gamma = \Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \cdots$$

and such that C_0 is C, $C_k \supseteq C_{k-1}$, and $C^* = \bigcup_k C_k$. Γ^* will be the union of the Γ_i's. Our construction must ensure that the Henkin model for Π^* is the same as the model $\mathcal{M}_{\Gamma^*}$.

Definition Let D be a set of constants and Λ be a set of D-sentences. Then $Th^{\varphi}[\Lambda, D]$ is the set

$$\{\theta : \theta \text{ is a semipositive } D\text{-sentence and } \Lambda \vdash_i \theta^{\varphi}\}$$

For us, the formula φ is fixed, so we also denote this set by $Th[\Lambda, D]$. If Δ is a classical theory then the $[\Lambda, D]$-*closure* of Δ is the classical theory axiomatized by $\Delta \cup Th[\Lambda, D]$.

Definition We define Γ_0 to be Γ, C_0 to be C and Π_0 to be the $[\Gamma, C]$-closure of T. For $k > 0$, Π_k, Γ_k and C_k are inductively defined by the following cases depending on the value of k mod 5:

(1) *Suppose* $k = 5j + 1$*:* Let $\Gamma_k = \Gamma_{k-1}$, let C_k be C_{k-1} plus the constant symbols occuring in α_j, and:

  (a) Let Π_k be $\Pi_{k-1} \cup \{\alpha_j\} \cup Th[\Gamma_k, C_k]$ if this theory is classically consistent,

  (b) Otherwise, let Π_k be $\Pi_{k-1} \cup \{\neg\alpha_j\} \cup Th[\Gamma_k, C_k]$

(2) *Suppose* $k = 5j + 2$*:* Let $\Gamma_k = \Gamma_{k-1}$. Note that by the previous case, α_j is a C_{k-1}-sentence. If α_j is of the form $(\exists x)\beta(x)$ and $\alpha_j \in \Pi_{k-1}$ then pick an arbitrary constant $c \in C^* \setminus C_{k-1}$ and define $C_k = C_{k-1} \cup \{c\}$ and let Π_k be the $[\Gamma_k, C_k]$-closure of $\Pi_{k-1} \cup \beta(c)$. Otherwise if α_j does not begin with an existential quantifier or is not in Π_{k-1}, let $C_k = C_{k-1}$ and $\Pi_k = \Pi_{k-1}$.

(3) *Suppose* $k = 5j + 3$*:* Let $C_k = C_{k-1}$. If α_j is of the form $\beta \vee \gamma$ and if $\Gamma_{k-1} \vdash_i \alpha_j$, then define as follows:

  (a) If the $[\Gamma_{k-1} \cup \{\beta\}, C_k]$-closure of Π_{k-1} is classically consistent then Π_k defined to be equal to this theory and Γ_k is $\Gamma_{k-1} \cup \{\beta\}$.

  (b) Otherwise, Γ_k is $\Gamma_{k-1} \cup \{\gamma\}$ and Π_k is the $[\Gamma_k, C_k]$-closure of Π_{k-1}.

  If, however, α_j is not a disjunction or is not an intuitionistic consequence of Γ_{k-1}, let $\Gamma_k = \Gamma_{k-1}$ and $\Pi_k = \Pi_{k-1}$.

(4) *Suppose* $k = 5j + 4$*:* If α_j is of the form $(\exists x)\beta(x)$ and if $\Gamma_{k-1} \vdash_i \alpha_j$, then pick an arbitrary constant $c \in C^* \setminus C_{k-1}$ and define $C_k = C_{k-1} \cup \{c\}$ and $\Gamma_k = \Gamma_{k-1} \cup \beta(c)$ and define Π_k to be the $[\Gamma_k, C_k]$-closure of Π_{k-1}. If, however, α_j does not begin with an existential quantifier or is not an intuitionistic consequence of Γ_{k-1}, let $\Gamma_k = \Gamma_{k-1}$ and $\Pi_k = \Pi_{k-1}$ and $C_k = C_{k-1}$.

(5) *Suppose* $k = 5j + 5$*:* Let $C_k = C_{k-1}$. If α_j is atomic and in Π_{k-1} then define $\Gamma_k = \Gamma_{k-1} \cup \{\alpha_j\}$ and Π_k to be the $[\Gamma_k, C_k]$-closure of Π_{k-1}. Otherwise if α_j is not atomic or is not in Π_{k-1}, let $\Gamma_k = \Gamma_{k-1}$ and $\Pi_k = \Pi_{k-1}$.

Define $\Pi^* = \bigcup_k \Pi_k$ and $\Gamma^* = \bigcup_k \Gamma_k$. Note $C^* = \bigcup_k C_k$.

The point of cases (1) and (2) above is to make Π^* a complete classical theory with witnesses for existential consequences. The point of cases (3) and (4) is to force Γ^* to be C^*-saturated. Case (5) serves to ensure that $\mathcal{M}_{\Gamma^*}$ will be equal to the Henkin model for Π^*. The requirement that Π_k contain $Th[\Gamma_k, C_k]$ serves to maintain the condition that $\Gamma_k \nvdash_i \varphi$.

Claim 1: For all $k \geq 0$,

(1) Π_k is classically consistent.

(2) $\Gamma_k \nvdash_i \varphi$ (so Γ_k is intuitionistically consistent).

Note that if $\Gamma_k \vdash_i \varphi$, then $\Gamma_k \vdash_i \bot^\varphi$ and hence $\Pi_k \vdash_c \bot$ and Π_k is inconsistent. So to prove the claim, it suffices to show Π_k is consistent which we do by induction on k. The base case is $k = 0$. Suppose for a contradiction that Π_0 is inconsistent. Then $T \vdash_c \neg\theta_1 \vee \neg\theta_2 \vee \cdots \vee \neg\theta_s$ for semipositive C-sentences θ_j such that $\Gamma \vdash_i \theta_j^\varphi$. By taking the conjunction of the θ_j's there is a single semipositive C-sentence θ such that $T \vdash_c \neg\theta$ and $\Gamma \vdash_i \theta^\varphi$. But it is immediate then by definition of the axioms of $\mathcal{H}T$ that $\mathcal{H}T \vdash_i \theta^\varphi \to \varphi$. Thus, since $\Gamma \supseteq \mathcal{H}T$, $\Gamma \vdash_i \varphi$; which is a contradiction.

For the induction step, we first assume Π_{k-1} is consistent and show that Π_k is consistent. We have five separate cases to consider.

Stage $k = 5j+1$. Let $\alpha_j = \alpha_j(\vec{e})$ where $\vec{e}$ are all the C^*-constants appearing in α_j that are not already in C_{k-1}. Let Π_k^a and Π_k^b be the $[\Gamma_k, C_k]$-closures of $\Pi_{k-1} \cup \alpha_j$ and $\Pi_{k-1} \cup \neg\alpha_j$, respectively. We need to show that at least one of these theories is classically consistent. Suppose both are inconsistent: then there are semipositive C_k-sentences $\theta_a(\vec{e})$ and $\theta_b(\vec{e})$, where all occurences of $\vec{e}$ are indicated, such that $\Gamma_{k-1} \vdash_i \theta_a(\vec{e})^\varphi$, $\Gamma_{k-1} \vdash_i \theta_b(\vec{e})^\varphi$, $\Pi_{k-1} \vdash_c \alpha_j(\vec{e}) \to \neg\theta_a(\vec{e})$ and $\Pi_{k-1} \vdash_c \neg\alpha_j(\vec{e}) \to \neg\theta_b(\vec{e})$. Then $\Pi_{k-1} \vdash_c \neg(\exists\vec{x})(\theta_a(\vec{x}) \wedge \theta_b(\vec{x}))$ and $\Gamma_{k-1} \vdash_i [(\forall\vec{x})(\theta_a(\vec{x}) \wedge \theta_b(\vec{x}))]^\varphi$. Since Π_{k-1} contains $Th[\Gamma_{k-1}, C_{k-1}]$, the semipositive C_{k-1}-sentence $(\forall\vec{x})(\theta_a(\vec{x}) \wedge \theta_b(\vec{x}))$ is in Π_{k-1}, contradicting the consistency of Π_{k-1}.

Stage $k = 5j+2$. Suppose $\alpha_j = (\exists x)\beta(x)$, $\Pi_{k-1} \vdash_c \alpha_j$, and $c \in C^* \setminus C_{k-1}$. If Π_k is inconsistent, there is a semipositive C_{k-1}-formula $\theta(x)$ such that $\Pi_{k-1} \vdash_c \beta(c) \to \neg\theta(c)$ and $\Gamma_{k-1} \vdash_i \theta(c)^\varphi$. Then, since c is a new constant symbol, $\Pi_{k-1} \vdash_c (\exists x)\beta(x) \to (\exists x)\neg\theta(x)$ and so $\Pi_{k-1} \vdash_c \neg(\forall x)\theta(x)$; also, $\Gamma_{k-1} \vdash_i [(\forall x)\theta(x)]^\varphi$. But $(\forall x)\theta(x)$ is a semipositive C_{k-1}-sentence and Π_{k-1}

contains $Th[\Gamma_{k-1}, C_{k-1}]$, so Π_{k-1} contains $(\forall x)\theta(x)$ which contradicts the induction hypothesis that Π_{k-1} is consistent.

Stage $k = 5j+3$. Suppose $\alpha_j = \beta \vee \gamma$ and $\Gamma \vdash_i \beta \vee \gamma$. Let Π_k^c and Π_k^d be the $[\Gamma_{k-1} \cup \{\beta\}, C_k]$-closure and $[\Gamma_{k-1} \cup \{\gamma\}, C_k]$-closure of Π_{k-1}, respectively. Suppose, for sake of a contradiction, that Π_k is inconsistent and hence both Π_k^c and Π_k^d are inconsistent. Then there are semipositive C_k-sentences θ_c and θ_d such that $\Gamma_{k-1} \vdash_i \beta \to (\theta_c)^\varphi$ and $\Gamma_{k-1} \vdash_i \gamma \to (\theta_d)^\varphi$ and such that $\Pi_{k-1} \vdash_c \neg\theta_c$ and $\Pi_{k-1} \vdash_c \neg\theta_d$. Since $\Gamma_{k-1} \vdash_i \alpha_j$, $\Gamma_{k-1} \vdash_i (\theta_c \vee \theta_d)^\varphi$ and hence the semipositive sentence $\theta_a \vee \theta_b$ is in Π_{k-1}. But this contradicts the induction hypothesis that Π_{k-1} is consistent.

Stage $k = 5j+4$. Suppose $\alpha_j = (\exists x)\beta(x)$, $\Gamma_{k-1} \vdash_i \alpha_j$ and $c \in C^* \setminus C_{k-1}$. If Π_k is inconsistent, then there is a semipositive C_{k-1}-sentence $\theta(x)$ such that $\Pi_{k-1} \vdash_c \neg\theta(c)$ and $\Gamma_{k-1} \vdash_i \beta(c) \to (\theta(c))^\varphi$. Since c is a new constant symbol, $\Pi_{k-1} \vdash_c (\forall x)\neg\theta(x)$ and likewise, since $\Gamma_{k-1} \vdash_i (\exists x)\beta_k(x)$, $\Gamma_{k-1} \vdash_i (\exists x)\theta(x)^\varphi$. Hence the semipositive C_{k-1}-sentence $(\exists x)\theta(x)$ is in Π_{k-1} which contradicts the consistency of Π_{k-1}.

Stage $k = 5j+5$. Suppose α_j is atomic and in Π_{k-1}. Suppose that Π_k is inconsistent. Then there is a semipositive C_k-sentence θ such that $\Pi_{k-1} \vdash_c \neg\theta$ and $\Gamma_{k-1} \vdash_i \alpha_j \to \theta^\varphi$. By the remark at the end of section 2, we also have $\Gamma_{k-1} \vdash_i (\alpha_j \vee \varphi) \to \theta^\varphi$; in other words, $\Gamma_{k-1} \vdash_i (\alpha_j \to \theta)^\varphi$. Hence the semipositive sentence $\alpha_j \to \theta$ is in Π_{k-1} and thus $\Pi_{k-1} \vdash_c \theta$, which contradicts the induction hypothesis that Π_{k-1} is consistent.

That completes the proof of Claim 1. □

Claim 2: If χ is atomic, then $\Gamma^* \vdash_i \chi$ if and only if $\Pi^* \vdash_c \chi$.

To prove Claim 2, first suppose that $\Gamma^* \vdash_i \chi$. Then for some k, χ is a C_k-sentence and $\Gamma_k \vdash_i \chi$; hence, $\Gamma_k \vdash_i \chi \vee \varphi$. In other words, $\Gamma_k \vdash_i \chi^\varphi$. Since Π_k contains $Th[\Gamma_k, C_k]$, $\Pi_k \vdash_c \varphi$. Conversely, if $\Pi^* \vdash_c \chi$ then some $\Pi_k \vdash_c \chi$. Now, for some large enough j, χ is equal to α_j and $\chi \in \Gamma_{5j+5}$. □

We are now ready to complete the proof of Lemma 4. Recall $\Gamma^* = \bigcup_k \Gamma_k$ and $\Pi^* = \bigcup_k \Pi_k$. By choice of constant symbols, $C^* = \bigcup_k C_k$. Clearly Π^* is a consistent, complete theory and since the α_j's enumerate all C^*-formulas, whenever $\Pi^* \vdash_c (\exists x)\beta(x)$ then $\Pi^* \vdash_c \beta(c)$ for some $c \in C^*$. Hence, the Henkin construction gives us a model $\mathcal{M}$ with domain a set of equivalence classes of C^* and $\mathcal{M} \vDash T$ since $\mathcal{M} \vDash \Pi^*$ and $\Pi^* \supset T$. Furthermore, since Γ^* and Π^* contain exactly the same atomic C^*-sentences, $\mathcal{M}_{\Gamma^*}$ is by definition

equal to the Henkin model $\mathcal{M}$. Thus condition (d) of Lemma 4 holds. Conditions (a)-(c) hold by the construction of Γ_k and by Claim 1.
Q.E.D. Lemma 4

We are now ready to define the T-normal Kripke model $\mathcal{K}$ for the proof of the completeness theorem. Recall that a T-normal Kripke model is an ordered pair $(\{\mathcal{M}_i\}_{i\in\mathcal{I}}, \preccurlyeq)$ where $\mathcal{I}$ is an index set, each $\mathcal{M}_i \vDash T$ and $\preccurlyeq$ is the reachability relation. The index set $\mathcal{I}$ will be the set of sets Γ of sentences such that

(a) Γ is C-saturated (where C is the set of constant symbols appearing in sentences in Γ,

(b) $\Gamma \supseteq \mathcal{H}T$, and

(c) $\mathcal{M}_\Gamma \vDash T$.

As an additional technical condition we require the set C of constant symbols be a coinfinite subset of some fixed countable set of constant symbols: this makes $\mathcal{I}$ a set rather than a proper class. Note that $\mathcal{I}$ is non-empty by Lemma 4. The worlds of $\mathcal{K}$ are the structures $\mathcal{M}_\Gamma$ such that $\Gamma \in \mathcal{I}$. By the Soundness Theorem proved above, $\mathcal{K} \Vdash \mathcal{H}T$. The reachability relation $\preccurlyeq$ is defined by $\mathcal{M}_{\Gamma_1} \preccurlyeq \mathcal{M}_{\Gamma_2}$ if and only if $\Gamma_1 \subseteq \Gamma_2$. It is easy to check from the definitions that, if $\Gamma_1 \subseteq \Gamma_2$ then $\mathcal{M}_{\Gamma_1}$ is a weak substructure of $\mathcal{M}_{\Gamma_2}$. Hence $\mathcal{K}$ is a T-normal Kripke model.

We are now ready to finish the proof of Theorem 1. Suppose $\mathcal{H}T \nvdash_i \varphi$ for φ an arbitrary sentence. By Lemma 4 there is a $\Gamma \in \mathcal{I}$ such that $\Gamma \nvdash_i \varphi$. It will suffice to prove that $\mathcal{M}_\Gamma \nVdash \varphi$ since then $\mathcal{K} \nVdash \varphi$. This follows from the next lemma which also implies that for any sentence θ, $\mathcal{H}T \vdash_i \theta$ if and only if $\mathcal{K} \Vdash \theta$; in other words, $\mathcal{K}$ is a T-normal, canonical Kripke model for $\mathcal{H}T$.

Lemma 5 *For any C-saturated $\Gamma \in \mathcal{I}$ and C-sentence ψ,*

$$\mathcal{M}_\Gamma \Vdash \psi \quad \Leftrightarrow \quad \Gamma \vdash_i \psi.$$

Proof (This is exactly like lemma 2.6.5 of Troelstra and van Dalen [4].) The lemma is proved by induction on the complexity of ψ:

Case (1): ψ is atomic. By definition of $\Vdash$ and $\mathcal{K}$.

Case (2): ψ is $\chi \wedge \gamma$.

$$\begin{aligned}\mathcal{M}_\Gamma \Vdash \chi \wedge \gamma &\Leftrightarrow \mathcal{M}_\Gamma \Vdash \chi \text{ and } \mathcal{M}_\Gamma \Vdash \gamma \\ &\Leftrightarrow \Gamma \vdash_i \chi \text{ and } \Gamma \vdash_i \gamma \qquad \text{by induction hypothesis} \\ &\Leftrightarrow \Gamma \vdash_i \chi \wedge \gamma\end{aligned}$$

Case (3): ψ is $\chi \vee \gamma$. Then

$$\begin{aligned}\mathcal{M}_\Gamma \Vdash \chi \vee \gamma &\Leftrightarrow \mathcal{M}_\Gamma \Vdash \chi \text{ or } \mathcal{M}_\Gamma \Vdash \gamma \\ &\Leftrightarrow \Gamma \vdash_i \chi \text{ or } \Gamma \vdash_i \gamma \qquad \text{by induction hypothesis} \\ &\Leftrightarrow \Gamma \vdash_i \chi \vee \gamma \qquad \text{by } C\text{-saturation of } \Gamma\end{aligned}$$

Case (4): ψ is $(\exists x)\chi(x)$. Then

$$\begin{aligned}\mathcal{M}_\Gamma \Vdash (\exists x)\chi(x) &\Leftrightarrow \exists c \in C,\ \mathcal{M}_\Gamma \Vdash \chi(c) \\ &\Leftrightarrow \exists c \in C,\ \Gamma \vdash_i \chi(c) \qquad \text{by induction hypothesis} \\ &\Leftrightarrow \Gamma \vdash_i (\exists x)\chi(x) \qquad \text{by } C\text{-saturation of } \Gamma\end{aligned}$$

Case (5): ψ is $\chi \rightarrow \gamma$. ($\Leftarrow$) First suppose $\Gamma \vdash_i \chi \rightarrow \gamma$. We must show that if $\mathcal{M}_\Gamma \preccurlyeq \mathcal{M}_{\Gamma_2}$ and $\mathcal{M}_{\Gamma_2} \Vdash \chi$ then $\mathcal{M}_{\Gamma_2} \Vdash \gamma$. Since $\Gamma_2 \supseteq \Gamma$, $\Gamma_2 \vdash_i \chi \rightarrow \gamma$. Hence, if $\mathcal{M}_{\Gamma_2} \Vdash \chi$ then, by the induction hypothesis $\Gamma_2 \vdash_i \chi$, so $\Gamma_2 \vdash_i \gamma$ and, again by the induction hypothesis, $\mathcal{M}_{\Gamma_2} \Vdash \gamma$. ($\Rightarrow$) Second suppose $\Gamma \nvdash_i \chi \rightarrow \gamma$. By Lemma 4, since $\Gamma \cup \{\chi\} \nvdash_i \gamma$, there is a $\mathcal{M}_{\Gamma_2} \succcurlyeq \mathcal{M}_\Gamma$ such that $\chi \in \Gamma_2$ and $\Gamma_2 \nvdash_i \gamma$. Now, by the induction hypothesis twice, $\mathcal{M}_{\Gamma_2} \Vdash \chi$ and $\mathcal{M}_{\Gamma_2} \nVdash \gamma$; so $\mathcal{M}_\Gamma \nVdash \chi \rightarrow \gamma$.

Case (6): ψ is $(\forall x)\chi(x)$. ($\Leftarrow$) First suppose $\Gamma \vdash_i (\forall x)\chi(x)$. Further suppose $\mathcal{M}_{\Gamma_2} \succcurlyeq \mathcal{M}_\Gamma$, Γ_2 is C_2-saturated and $c \in C_2$. Then $\Gamma_2 \vdash_i \chi(c)$ since $\Gamma_2 \supseteq \Gamma$ and by the induction hypothesis, $\mathcal{M}_{\Gamma_2} \Vdash \chi(c)$. Hence $\mathcal{M}_\Gamma \Vdash (\forall x)\chi(x)$. ($\Rightarrow$) Second suppose $\Gamma \nvdash_i (\forall x)\chi(x)$. If c is a new constant symbol not in C, then $\Gamma \nvdash_i \chi(c)$. By Lemma 4 there is a world $\mathcal{M}_{\Gamma_2} \succcurlyeq \mathcal{M}_\Gamma$ such that $\Gamma_2 \nvdash_i \chi(c)$ with c a constant symbol in the language of Γ_2. Now by the induction hypothesis, $\mathcal{M}_{\Gamma_2} \nVdash \chi(c)$ so $\mathcal{M}_\Gamma \nVdash (\forall x)\chi(x)$.

Q.E.D. Lemma 5 and the Completeness Theorem.

4 Examples

4.1 Pure first-order logic.

Pure first-order logic is a first-order theory in an arbitrary language with no non-logical axioms. Let T be pure classical logic in some language. Of course, any Kripke structure is T-normal. Hence the above soundness and completeness theorem combined with the usual soundness and completeness theorems for intuitionistic logic and Kripke structures immediately imply that $\mathcal{H}T$ is the intuitionistic version of pure first-order logic.

Another way to prove this is to use the definition of $\mathcal{H}T$ and Lemma 3.5.3(i) of Troelstra-van Dalen [4].

4.2 Peano arithmetic.

We shall next consider Peano arithmetic (PA) with the language of PRA, that is, containing nonlogical symbols for every primitive recursive function and relation. Because of the choice of language, $\mathcal{H}PA$ proves the law of the excluded middle for atomic (primitive recursive) formulas. To see this, let $R(\vec{x})$ be atomic. There is a predicate symbol Q such that $PA \vdash_c R(\vec{c}) \leftrightarrow \neg Q(\vec{x})$. In an PA-normal Kripke structure $\mathcal{K}$ and in any world $\mathcal{M}$ of $\mathcal{K}$, for any $\vec{c} \in |\mathcal{M}|$, we claim that

$$\mathcal{M} \Vdash R(\vec{c}) \vee \neg R(\vec{c}).$$

This is since if $R(\vec{c})$ is classically true in $\mathcal{M}$, then $\mathcal{M} \Vdash R(\vec{c})$ by definition. Otherwise, $Q(\vec{c})$ is classically true in $\mathcal{M}$ and every world accessible from $\mathcal{M}$. Hence $R(\vec{c})$ can not true in any world accessible from $\mathcal{M}$, so $\mathcal{M} \Vdash \neg R(\vec{c})$.

Alternatively, the law of excluded middle for primitive recursive properties can be proof-theoretically established by considering the $\mathcal{H}T$ axiom

$$(\neg R \vee \neg Q)^{R \vee Q} \to (R \vee Q).$$

Theorem 6 *$\mathcal{H}PA$ is a subtheory of HA.*

Proof This is because of the fact that if $PA \vdash_c \neg\theta$ then $HA \vdash_i \theta^{\varphi} \to \varphi$ where θ is a semipositive formula and φ is an arbitrary formula.[‡] □

[‡]This corrects the proof in the journal version. I appreciate K. Wehmeier's pointing out the error to me.

In a moment we shall see that $\mathcal{H}PA$ does not equal Heyting arithmetic HA; but first we prove that $\mathcal{H}PA$ does admit induction on existential formulas. By 'existential formula' we mean a formula with a prefix of existential formulas and no other quantifiers.

Theorem 7 *If $A(z)$ is an existential formula then $\mathcal{H}PA$ proves*

$$A(0) \wedge (\forall z)(A(z) \to A(z+1)) \to (\forall z)A(z).$$

Proof We shall give a model-theoretic proof. Suppose A is a formula of the form $(\exists\vec{x})R(z,x,\vec{b})$ where $\vec{b}$ are parameters and R is w.l.o.g. atomic. Further suppose $\mathcal{K}$ is a Kripke structure, $\mathcal{M}_0$ is a world in $\mathcal{K}$, $\vec{m}$ are members of the universe of $\mathcal{M}_0$, and

$$\mathcal{M}_0 \Vdash (\exists\vec{x})R(0,\vec{x},\vec{m}) \wedge (\forall z)[(\exists\vec{x})R(z,\vec{x},\vec{m}) \to (\exists\vec{x})R(z+1,\vec{x},\vec{m})].$$

We must show that if $\mathcal{M}$ is reachable from $\mathcal{M}_0$ and if $c \in |\mathcal{M}|$ then $\mathcal{M} \Vdash (\exists\vec{x})R(c,\vec{x},\vec{m})$. It is obvious from the definition of forcing that a existential formula is intuitionistically forced in $\mathcal{M}$ if and only if it is classically true in $\mathcal{M}$. Thus, we have

$$\mathcal{M} \models (\exists\vec{x})R(0,\vec{x},\vec{m}) \wedge (\forall z)[(\exists\vec{x})R(z,\vec{x},\vec{m}) \to (\exists\vec{x})R(z+1,\vec{x},\vec{m})].$$

Hence, since $\mathcal{M} \models PA$, $(\exists\vec{x})R(c,\vec{x},\vec{m})$ is classically true and intuitionistically forced in $\mathcal{M}$. $\square$

A proof-theoretic proof of Proposition 7 can be obtained by letting θ be the semipositive formula

$$(\exists\vec{x})R(0,\vec{x},\vec{b}) \wedge (\forall z)(\forall\vec{x}_1)[R(z,\vec{x}_1,\vec{b}) \to (\exists\vec{x}_2)R(z+1,\vec{x}_2,\vec{b})] \wedge (\forall x)\neg R(c,\vec{x},\vec{b})$$

and letting φ be $(\exists x)R(c,\vec{x},\vec{b})$. Now θ is essentially the negation of the induction axiom for $A(z)$ and $T \vdash_c \neg\theta$. Hence $\mathcal{H}T$ has $\theta^\varphi \to \varphi$ as an axiom and it can be easily seen that this axiom intuitionistically implies the induction axiom for $A(z)$.

Theorem 8 $\mathcal{H}PA \neq HA$.

To prove Theorem 8 we will show that $\mathcal{H}PA$ does not prove induction on Π_1-formulas (universal formulas). First, we need to develop a variant of

Gödel's second incompleteness theorem in the next two lemmas. If n is an integer, then $\underline{n}$ is a canonical closed term with value n; if ψ is a formula with Gödel number n, then $\ulcorner\psi\urcorner$ denotes the term $\underline{n}$. For $n > 0$, $I\Sigma_n$ is the subtheory of PA with induction restricted to Σ_n-formulas (since our language is the language of PRA, we refrain from defining $I\Sigma_0$ here). Recall that arithmetization of metamathematics can be carried out in $I\Sigma_1$; for example, there is a sentence $Con(PA)$ which expresses the consistency of Peano arithmetic and there is a formula $Thm_{PA}(w)$ which states that w is the Gödel number of a theorem of PA. Likewise there are formulas $Con(I\Sigma_a)$ and $Thm_{I\Sigma_a}(w)$: note especially that a is a free variable in these formulas. The usual Hilbert-Bernay derivability conditions hold, in particular, if ψ is an existential sentence then $I\Sigma_1 \vdash_c \psi \rightarrow Thm_{I\Sigma_1}(\ulcorner\psi\urcorner)$.

Lemma 9 is well-known and provides motivation for Lemma 10.

Lemma 9 *Let T be PA or be one of the theories $I\Sigma_i$. If ψ is a existential sentence consistent with T then*

$$T + \psi \nvdash_c Con(T).$$

Proof Since ψ is existential, $T + \psi \vdash_c Thm_T(\ulcorner\psi\urcorner)$. Hence,

$$T + \psi \vdash_c Con(T) \rightarrow Con(T + \psi).$$

But by Gödel's second incompleteness theorem, $T + \psi \nvdash_c Con(T + \psi)$, so $T + \psi \nvdash_c Con(T)$. $\Box$

For the next lemma, let the language of Peano arithmetic be enlarged to include a new uninterpreted constant symbol a.

Lemma 10 *Let $\psi(a)$ be an existential sentence and T be the theory $PA \cup \{a > \underline{n} : n \geq 0\} \cup \psi(a)$. If T is consistent, then $T \nvdash_c Con(I\Sigma_a)$.*

Proof (by contradiction.) Suppose $T \vdash_c Con(I\Sigma_a)$. By compactness, there is an integer k such that

$$I\Sigma_k \cup \{a > \underline{k}, \psi(a)\} \vdash_c Con(I\Sigma_a).$$

Hence, $I\Sigma_k \cup \{a > \underline{k}, \psi(a)\} \vdash_c Con(I\Sigma_k)$. From this it follows immediately that

$$I\Sigma_k + (\exists x)(x > \underline{k} \wedge \psi(x)) \vdash_c Con(I\Sigma_k);$$

which contradicts Lemma 9. $\Box$

We are now ready to prove Theorem 8. We shall construct a PA-normal Kripke model $\mathcal{K}$ in which induction fails for the Π_1-formula $Con(I\Sigma_a)$. The worlds of $\mathcal{K}$ will be models $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \ldots$ of $PA \cup \{\neg Con(PA)\}$ such that $\mathcal{M}_i \preccurlyeq \mathcal{M}_j$ if and only $i \leq j$. Of course, in any such model $\mathcal{M}_i$ there is a least (necessarily nonstandard) number a_i such that $\neg Con(I\Sigma_{a_i})$ is classically true in $\mathcal{M}_i$. Pick $\mathcal{M}_1$ to be an arbitrary countable model of $PA \cup \{\neg Con(PA)\}$. Now assume $\mathcal{M}_i$ has been chosen; we describe how to pick $\mathcal{M}_{i+1}$. Let $\Delta(\mathcal{M}_i)$ be the atomic diagram of $\mathcal{M}_i$; that is to say, the theory with constant symbols for each member of $|\mathcal{M}_i|$ containing all atomic sentences true in $\mathcal{M}_i$. Let b be a new constant symbol. Then $\mathcal{M}_{i+1}$ is to be an arbitrary countable model of

$$PA \cup \Delta(\mathcal{M}_i) \cup \{b < a_i, \neg Con(I\Sigma_b)\}.$$

Of course, we must show this theory is consistent. Suppose it is inconsistent: then, by compactness, there is a quantifier-free formula $\delta(\vec{d}, a_i)$ true in $\mathcal{M}_i$ such that

$$PA \cup \{\delta(\vec{d}, a_i), b < a_i\} \vdash_c Con(I\Sigma_b).$$

From this we have

$$PA \cup \{(\exists \vec{x})(\exists y)(\delta(\vec{x}, y) \wedge b < y)\} \vdash_c Con(I\Sigma_b).$$

But this is impossible by Lemma 10 since the theory

$$PA \cup \{(\exists \vec{x})(\exists y)(\delta(\vec{x}, y) \wedge c < y)\} \cup \{c > \underline{m} : m \geq 0\}$$

is consistent since it has $\mathcal{M}_i$ as a model (with $c = a_i - 1$).

By construction, we have $a_1 > a_2 > a_3 > \cdots$ and furthermore it follows directly from the definition of forcing that, for any j and any $a \in |\mathcal{M}_i|$, we have $\mathcal{M}_i \Vdash Con(I\Sigma_a)$ if and only if $a < a_i$ for all i. Then $\mathcal{K} \Vdash Con(I\Sigma_1)$ and $\mathcal{K} \Vdash (\forall x) Con(I\Sigma_x) \to Con(I\Sigma_{x+1}))$ but $\mathcal{K} \nVdash Con(I\Sigma_{a_1})$. In other words, induction on the formula $Con(PA_a)$ with respect to a is not intuitionistically forced in the PA-normal Kripke model PA.

Q.E.D. Theorem 8

We conclude with a few questions. Firstly, is every intuitionistic theory equivalent to a theory $\mathcal{H}T$ for some classical theory T? If not, is there a natural characterization of the intuitionistic theories of the form $\mathcal{H}T$? In particular, is there a classical theory T such that HA is equal to $\mathcal{H}T$?

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