Ed Nelson's Work on Logic and Foundations: Radical Constructivism

Sam Buss
Department of Mathematics
U.C. San Diego

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I. Nelson's philosophy: a radical form of constructivism

Platonists believe in the full, independent existence our usual mathematical constructs, including integers, reals, the powerset of the reals, even abstract sets. Ordinary constructivists, in the spirit of D. Hilbert, accept use of the completed infinity of integers, the use of primitive recursive functions, etc.

Nelson [PA, p.1]: **The reason for mistrusting the induction principle is that it involves an impredicative concept of number.** It is not correct to argue that induction only involves the numbers from 0 to n; then property of n being established may be a formula with bound variables that are thought of as ranging over all numbers. **That is, the induction principle assumes that the natural number system is given.** A number is conceived to be an object satisfying every inductive formula; for a particular inductive formula, therefore, the bound variables are conceived to range over objects satisfying every inductive formula, including the one in question. *(emphasis added)*

Doubt About the Integers?

How could one doubt the integers? Even without believing in the integers as “physical” entities, one surely should believe in them as a set of mental constructs that have definite properties.

For example: **Do there exist odd perfect integers?**

Whether there exists an odd perfect integer should be a definite property of the integers. That is, they exist, or do not exist, independently of the successes or failures human efforts in doing mathematics.

In contrast, reasonable people might agree to doubt the relevance or the meaning of the continuum hypotheses CH or GCH. There could be multiple, equally compelling concepts of “set” and thus no reason to believe that CH or GCH have any independent meaning as a platonic truth (or platonic falsity).

Engendering Doubt About the Integers

Suppose, hypothetically, that you believe beyond doubt in the integers and the existence of the set of all integers in some sense.

Well, the concept of a *set of integers* also surely makes sense. So consider the class of all sets of integers, that is, the powerset, $P(\mathbb{N})$, of the integers. Now, consider the continuum hypothesis question: Is there a subset of $P(\mathbb{N})$ of cardinality strictly between the cardinalities of the $\mathbb{N}$ and $P(\mathbb{N})$? This is just a question about relations on $P(\mathbb{N}) \times P(\mathbb{N})$. — That is, it is a question about sets of pairs (binary relations) which are subsets of $P(\mathbb{N}) \times P(\mathbb{N})$.

But then you must believe (beyond doubt) in the meaningfulness of the continuum hypothesis problem. And this was something you might think reasonable to doubt.

The above reasoning gives us three options:

1. A platonic belief in the existence of integers and reals, including belief in the meaningfulness of the continuum hypothesis.

2. A platonic belief in the existence of the set of integers, i.e., in the completed infinity; but doubt about the meaningfulness of forming powersets of infinite sets.

3. Doubt about the existence of the set of all integers.

Option **1.** is the traditional mathematical viewpoint of course, but the history of set theory is not encouraging.
Is **3.** any more unreasonable than **2.**?

There is one more option: but it is the coward's way out:
4. An agnostic viewpoint which refuses to worry about the issue.

II. Nelson's Predicative Arithmetic

Hilbert's program: Hilbert suggested that as a first step before considering "truth" or "semantics", one should consider "syntax" of proofs and the "consistency" of theories.

Hilbert's program for establishing consistency was foiled by Gödel's incompleteness theorems already at the level of formal theories of arithmetic. Nelson's radical constructivism takes this failure at face value and doubts even the consistency of commonly used theories of the integers!

Welcome fact: Even if one does not buy into radical constructivism, there is still interesting mathematics to do with weak systems of consistency strength much weaker than Peano arithmetic.

Predicative arithmetic: a weak form of arithmetic which does not make platonic assumptions about the existence of a completed infinity of integers.

Base Language and Axioms of Predicative Arithmetic

First-Order Logic: $\wedge$, $\vee$, $\neg$, $\rightarrow$, $\forall$, $\exists$, $=$

Fucntion Symbols: 0, S (successor), $+$ (addition), $\cdot$ (multiplication).

Robinson's Theory Q:

$$\begin{array}{ll} Sx \neq 0 & x + 0 = x \\ Sx = Sy \rightarrow x = y & x + Sy = S(x + y) \\ x \neq 0 \rightarrow (\exists y)(Sy = x) & x \cdot 0 = 0 \\ & x \cdot Sy = x \cdot y + x. \end{array}$$

Q is very weak, but still subject to Gödel's incompleteness theorem. Q does not even prove $x + y = y + x$ or $0 + x = x$.

Q does *not* include any induction axioms. It is a very weak subtheory of Peano arithmetic (PA). For convenience, include associativity of $+$, $\cdot$ in Q.

Extensions by Definition

The theory Q can be conservatively extended by adding new symbols defined in terms of old ones. For example:

Inequality Predicate:

$$x \leq y \;\Leftrightarrow\; (\exists z)(x + z = y)$$

Predecessor Function:

$$P(x) = y \;\Leftrightarrow\; S(y) = x \vee (x = 0 \wedge y = 0).$$

Extension by definitions gives a conservative extension of Q, so they can be used freely.

The consistency of assuming $0 + x = x$

Def'n A formula $\phi(x)$ is inductive provided it has been proved that

$$\phi(0) \;\wedge\; \forall x(\phi(x) \to \phi(Sx)).$$

Theorem: (Solovay[unpub.],Nelson[PA]) Suppose $\phi(x)$ is inductive. Let

$\phi^1(x)$ be $\forall y(y \leq x \to \phi(y))$.
$\phi^2(x)$ be $\forall y(\phi^1(y) \to \phi^1(y + x))$.
$\phi^3(x)$ be $\forall y(\phi^2(y) \to \phi^2(y \cdot x))$.

Then, $\phi^3(x)$ defines an initial segment of the integers which is closed under S, $+$ and $\cdot$.

Thm: Let $\psi(x)$ be the formula $0 + x = x$. Then, $\psi(x)$ is inductive.

Pf: (a) $0 + 0 = 0$ holds by an axiom of Q.
(b) Suppose $0 + x = x$. Then $0 + Sx = S(0 + x) = Sx$. $\square$.

Thus $\{x : \psi^3(x)\}$ is a set of integers that satisfies $Q_2 := Q + \forall x(0 + x = x)$.

This has provided an "interpretation" of Q_2 in Q.

Semantic viewpoint: starting with a mass of 'integers' that satisfy Q, we have found an initial segment of integers that also satisfies $\forall x(0 + x = x)$.

Syntactic viewpoint: if Q is consistent, then Q_2 is consistent. Using 'relativization', any Q_2-proof can be transformed into a Q-proof. By relativization is meant restricting attention to integers that satisfy $\psi^3(x)$.

Def'n [Nelson, PA] A theory $T \supset Q$ is predicative if it is interpretable in Q. This include allows multiple uses of extension by definition and of interpretation with initial segments obtained from inductive formulas.

Examples of predicative principles: [Nelson, PA]

1. Induction on bounded formulas. Bounded formulas may only use quantifiers which are bounded, $\forall x < t$ and $\exists x < t$.
2. Least number principles for bounded formulas.
3. Sequence coding, Gödel numbers for syntactic objects including formulas and proofs. The smash function, $\#$,

$$x \# y = 2^{|x| \cdot |y|}$$

where $|x| \approx \log_2(x)$. Metamathematic concepts including consistency and interpretability and the proof of the Gödel incompleteness theorem.

Some principles which are **not** predicative include:

1. The totality of exponentiation: *exp*$:=\forall x \exists y (2^x = y)$.

2. Having an inductive initial segment on which superexponentiation $2 \Uparrow x$ is total. Here, $2 \Uparrow 0 = 1$ and $2 \Uparrow (x+1) = 2^{2 \Uparrow x}$.

3. The Gentzen cut elimination theorem.

4. The consistency of the theory Q.

However, w.r.t. **1.**, principles that follow from a *finite* number of uses of exponentiation are predicative. E.g., the tautological consistency of Q and the bounded consistency of Q. Futhermore, Wilkie-Paris showed that any bounded formula which is a consequence of $Q +$ *exp* is predicative.

Computational Complexity & More Predicative Constructions

Bounded Arithmetic: Essentially Q plus induction for all bounded formulas. My formulation of the S_2^i and T_2^i theories of bounded arithmetic borrowed from Nelson's treatment (esp., the use of smash), and from the work of Dimitracopolous, Paris, Wilkie, Pudlák (who use the Ω_1 axiom instead).

Theories of bounded arithmetic are closely connected to feasible complexity classes and related classes. For instance, S_2^1 has proof strength equivalent to polynomial time computability.

The classes of P (polynomial time), NP (nondeterministic polynomial time) and the polynomial hierarchy are all predicative. That is, all functions and predicates in these classes can be predicatively introduced.

Thm:
(a) The polynomial space (PSPACE) predicates can all be introduced predicatively.

(b) The exponential time (EXPTIME) predicates can all be introduced predicatively.

Proof idea: Introduce predicate symbols that represent a PSPACE (or EXPTIME) predicate using defining axioms that implicitly define the predicate. For EXPTIME this requires the Chandra-Kozen-Stockmeyer charactization in terms of alternating polynomial space. These predicate symbols are definable explicitly in the presence of one exponential; then define an inductive initial segment on which one use of exponentiation is available. □

Thm: Real analysis, up to at least standard theorems on integration can be developed predicatively.

Proof ideas: Jay Hook, in his 1983 Ph.D. thesis does this under the assumption that exponentiation is not total. The assumption can be removed.

In principle, the work of K. Ko and H. Friedman also indicates a way to predicatively develop real analysis since they show integration is in PSPACE (but Ko and Friedman only consider computability, not provability).
See also recent work of Fernando and Ferreira on formulations of real analysis in bounded arithmetic.

Synopsis of Predicative Arithmetic

- Begin with a underspecified mass of integers that are closed under successor, addition and multiplication. The latter two closure properties could be replaced by ternary relations; only the assumption of the successor operation is needed. Motivation: models the integers that can be written in unary notation, the integers that can be counted to.
- Criticism: Why should we accept the above as an infinite set? Isn't such acceptance implicit in the use of unbounded quantifiers in ϕ^2 and ϕ^3?
- Using interpretations, especially inductive initial segments, develop a more refined concept of integer. A compellingly effective treatment of much basic mathematics has been done.
- Nelson asked a "compatibility problem" question: If A and B are predicative principles, then must $A \wedge B$ also be predicative? Solovay showed the answer is no, but with a non-appealing example. Open Question: Are there nonetheless useful compatibility results? (If nothing else, $Q+A\wedge B$ is always consistent platonically.)

III. Nelson's Automated Proof Checker

In an unpublished (unfinished?) 1993 manuscript, [Nelson,NT] revisits the develop of predicative arithmetic, with a automated proof checker.

He introduces an automated proof checker, *qed*, that is incorporated in his LaTeX files. Theorems are stated and proved in a formal system that is automatically checked by the computer. The technical content of the theorems is similar to [Nelson, PA].

The proof system *qed* is a kind of deduction proof system (similar to a deduction proof system of Fitch, but using very different notations). To illustrate, consider using the axiom

$$\forall x(x + Sy = S(x + y) \tag{1}$$

to prove

$$x = 0 + x \to Sx = 0 + Sx \tag{2}$$

Written out in full the proof looks like:

$\forall x(x + Sy = S(x + y)$	(1) Hypothesis (Axiom)
$x = 0 + x \rightarrow Sx = 0 + Sx$	(2) Goal to be proved
$\{$	Assume its negation
$\quad e = 0 + e \wedge Se \neq 0 + Se$	(3) New variable e for x in $\neg(2)$.
$\quad 0 + Se = S(0 + e)$	(4) Instance of (1).
$\}$	Simple contradiction reached.

By "simple contradiction" is meant a polynomial time test for tautological unsatisfiability. More general nesting of assumptions of (negations of) goals is permitted.

Compact representation of the above proof: (x used in place of e).

$$2\{:x\ 1;0;x\}$$

From [Nelson, NT]:

Th 158: $x \neq 0 \quad \rightarrow \quad x/x = 1.$

158{:x 113;x;x;1;0 16;x 47;x 130;x 3;$x\cdot 1$ 134;x }.

Th 159: $x_1 \leq x_2 \quad \rightarrow \quad x_1/y \leq x_2/y.$

We have (.1) $y \neq 0$. There is a non-zero u such that $x_2/y + u = x_1/y$, so $x_1 = y \cdot (x_2/y + u) + r_1 = \big((y \cdot (x_2/y) + (y \cdot u)\big) + r_1 = y \cdot (x_2/y) + (y \cdot u + r_1)$. There is a z such that $x_2 = x_1 + z$, so that $x_2 = \big(y \cdot (x_2/y) + (y \cdot u + r_1)\big) + z = (y \cdot (x_2/y) + \big((y \cdot u + r_1) + z\big)$. Consequently, $r_2 = (y \cdot u + r_1) + z = y \cdot u + (r_1 + z)$, so $y \cdot u \leq r_2$ and hence $y \cdot u < y$, which is impossible.

159{:x_1:x_2:y .1{ 156;x_1 156;x_2 16;0 } 113;x_1;y;x_1/y:r_1 113;x_2;y;x_2/y:r_2 98;x_2/y;x_1/y 44;x_2/y;x_1/y:u 10;y;x_2/y;u 9;$y\cdot(x_2/y)$;$y\cdot u$;r_1 15;x_1;x_2:z 9;$y\cdot(x_2/y)$;$y\cdot u{+}r_1$;z 9;$y\cdot u$;r_1;z 54;$y\cdot(x_2/y)$;$(y\cdot u{+}r_1){+}z$;r_2 14;$y\cdot u$;$r_1{+}z$;r_2 69;$y\cdot u$;r_2;y 95;y;u }.

[Nelson, NT, p.88-89]:

> It must be exhilarating to the superbly skilled people restoring the Sistine Chapel to reveal the original work that lay under the smoke and grime of centuries. I felt exhilaration writing Chapter 2: for the first time I experienced mathematics without the obscuring layer of semantics.
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> ... I feel confident now that complete formalization of mathematics is not only feasible, but practical. The question remains: is it worthwhile? To me the answer is clearly yes.

[Nelson, NT, p.89]:

> ... In the not distant future there will be huge data banks of theorems with rapid search procedures to help mathematicians construct proofs of new theorems. ...
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> But for centuries to come, human mathematicians will not be replaced by computers. We have different search skills. There is a phase transition separating feasible searches from infeasible ones, a phase transition that is roughly described by the distinction between polynomial time algorithms and exponential time algorithms. The latter are in general infeasible; they will remain forever beyond the reach of both people and machines.

IV. Concluding Thoughts

Nelson [M&F, p.7]

> Now I live in a world in which there are no numbers save those that human being on occasion construct.

Nelson [M&F, p.4]

> Mathematicians no more discover truths than the sculpter discovers the sculpture inside the stone. [...] But, unlike sculpting, our work is tightly constrained, both by the strict requirements of syntax and by the collegial nature of the enterprise. This is how mathematics differs profoundly from art.

Nelson [PA, p.50]:

> Perhaps infinity is not far off in space or time or thought; perhaps it is while engaged in an ordinary activity—writing a page, getting a child ready for school, talking with someone, teaching a class, making love—that we are immersed in infinity.

Sources for quotes

[PA] E. Nelson, Predicative Arithmetic, Princeton University Press, 1986.

[M&F] E. Nelson, Mathematics and Faith. Available at http://www.math.princeton.edu
Presented at the Jubilee for Men and Women from the World of Learning, The Vatican, 23-24 May 2000.

[NT] E. Nelson, untitled, unpublished manuscript, 1993.